CONVEX RELAXATION STRATEGIES FOR COMPUTER GRAPHICS

Leticia Mattos Da Silva

Massachusetts Institute of Technology

OUR ROADMAP

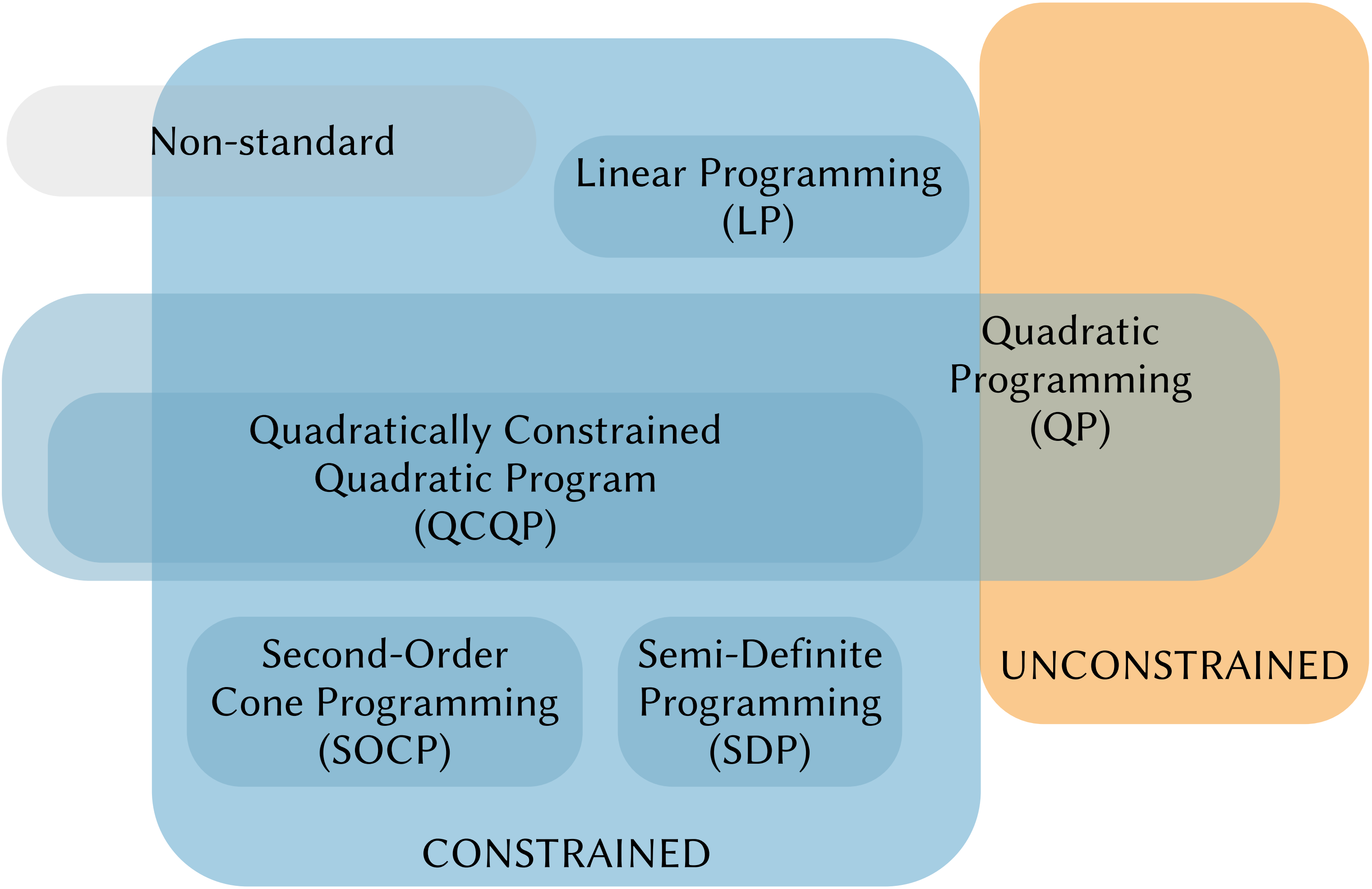
OPTIMIZATION

1. The Fundamentals
2. Convexity
3. Why Convexity?
4. Standard Convex Problems

CONVEX RELAXATION

5. Viscosity Solutions
6. Mapping Problems
7. Optimal Transport
8. SOS Relaxations
9. Convex Substructures

SIMPLIFIED TAXONOMY OF CONVEX



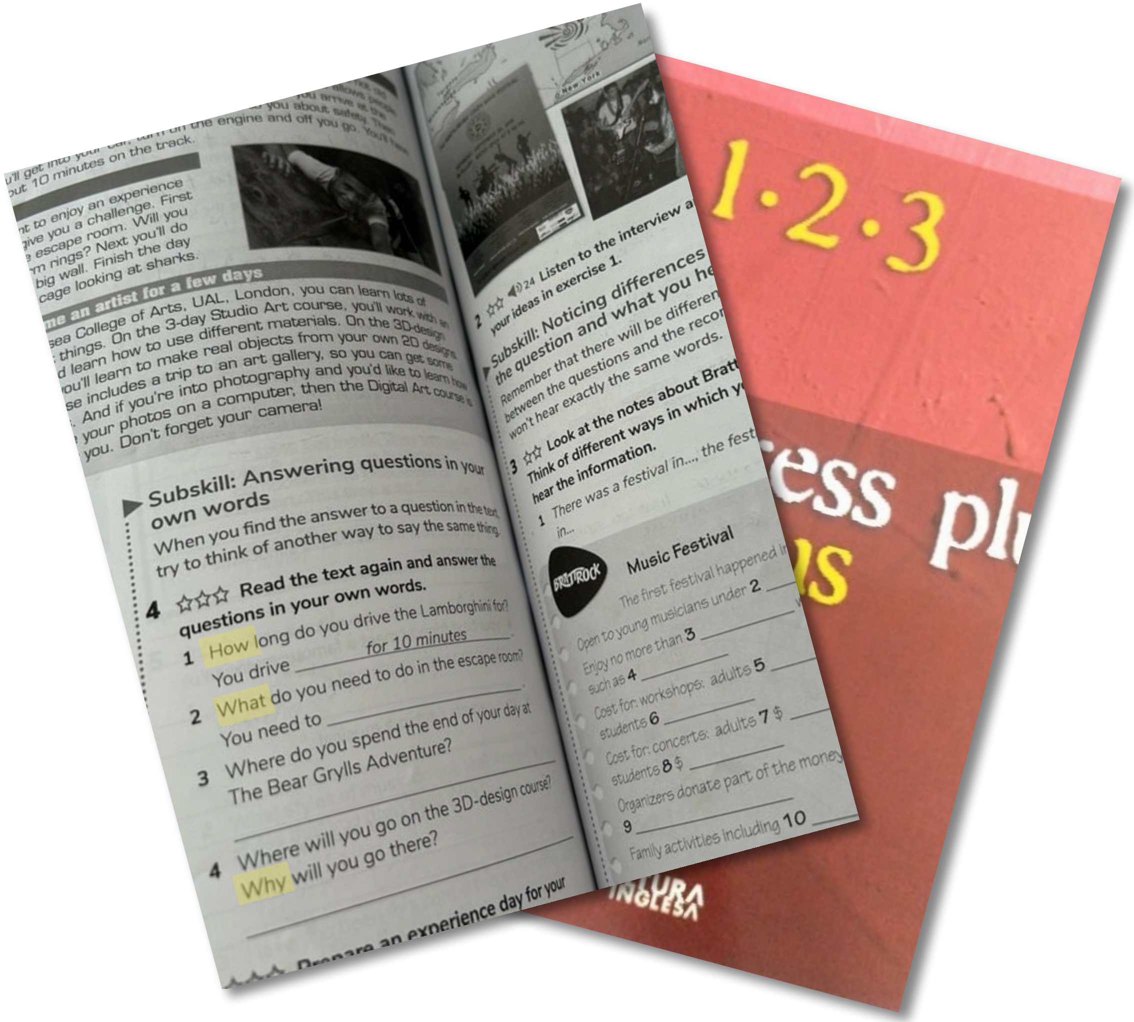
OUR END GOAL

💡 let's use question words!

WHY
Why convex relaxation?

WHAT
What can be done via convex relaxation?

HOW
How is convex relaxation used?



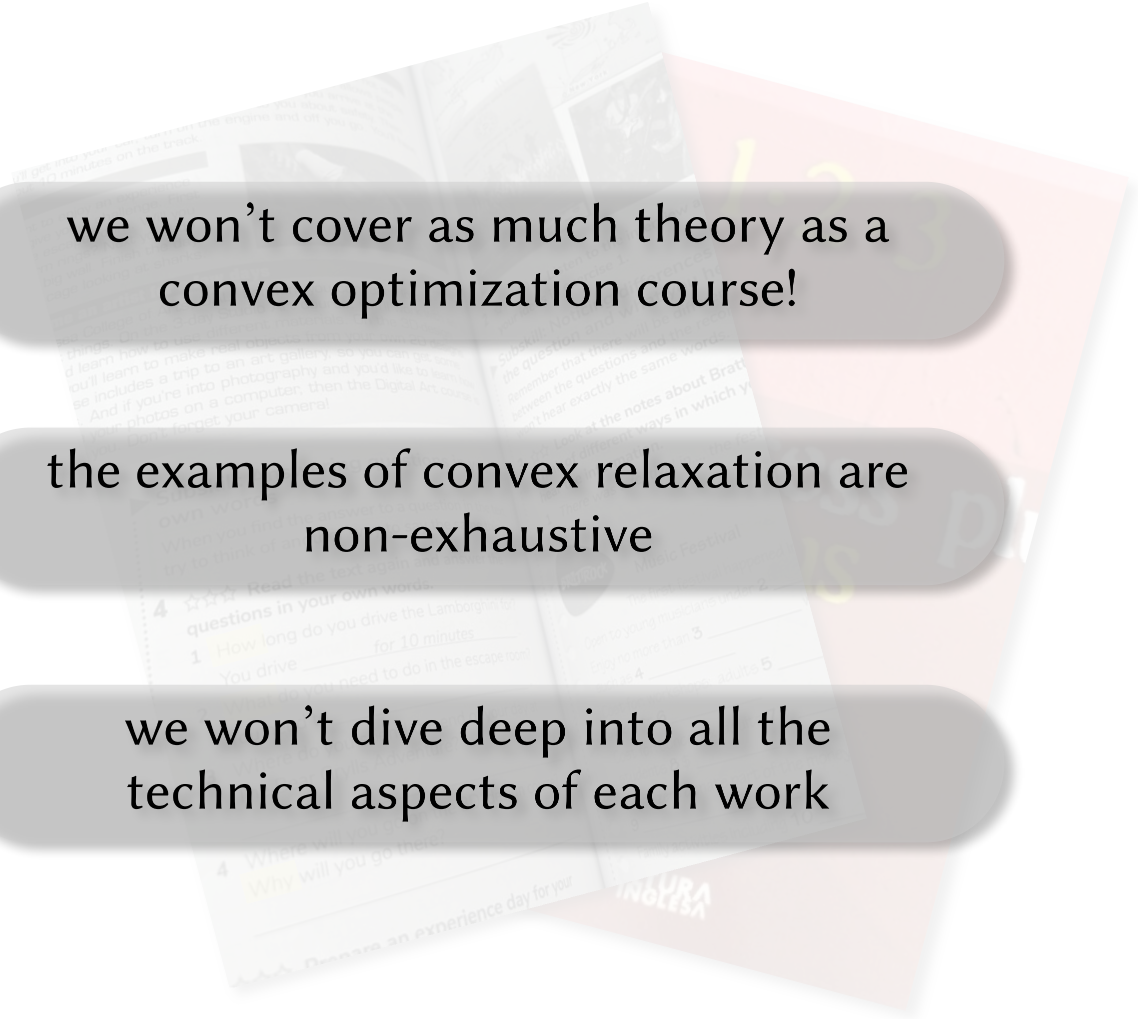
OUR END GOAL

💡 let’s use question words!

WHY
Why convex relaxation?

WHAT
What can be done via convex relaxation?

HOW
How is convex relaxation used?



we won’t cover as much theory as a convex optimization course!

the examples of convex relaxation are non-exhaustive

we won’t dive deep into all the technical aspects of each work

OUR ROADMAP

OPTIMIZATION

1. The Fundamentals
2. Convexity
3. Why Convexity?
4. Standard Convex Problems

CONVEX RELAXATION

5. Viscosity Solutions
6. Mapping Problems
7. Optimal Transport
8. SOS Relaxations
9. Convex Substructures

SIMPLIFIED TAXONOMY OF CONVEX

THE FUNDAMENTALS

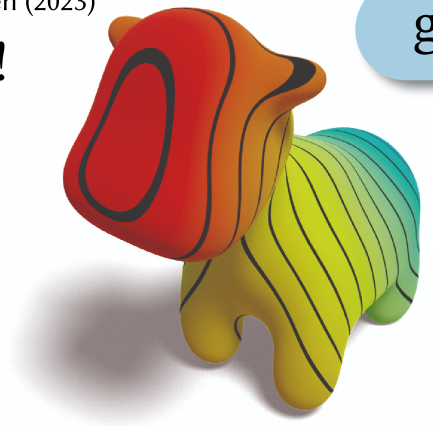
Computer graphics is full of hard math problems!

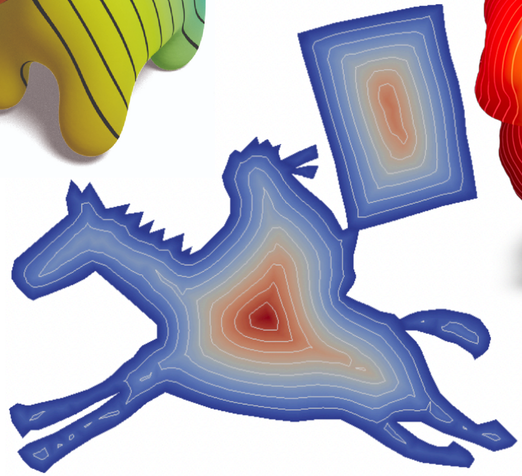
shape correspondence

Abdelreheem, Eldesokey, Ovsjanikov, Wonka (2023)

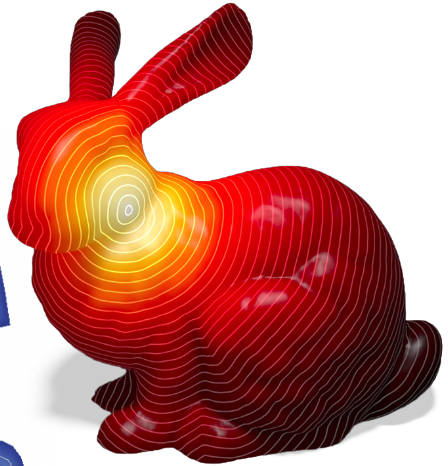
geodesic distances

Edelstein, Guillén, Solomon, Ben-Chen (2023)





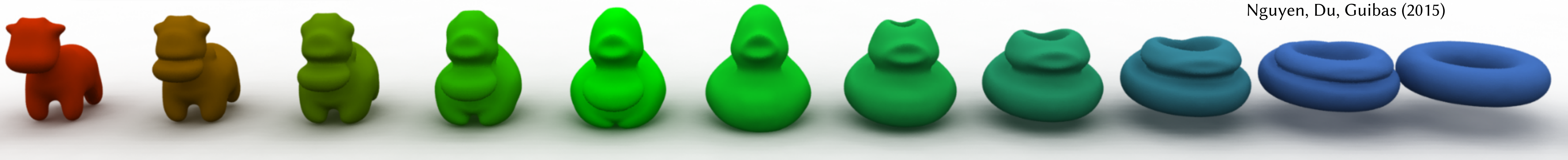
Belyaev & Fayolle (2015)



Crane, Weischedel, Wardetzky (2013)

optimal transport

Solomon, Goes, Peyre, Cuturi, Butscher, Nguyen, Du, Guibas (2015)



THE FUNDAMENTALS

Computer graphics is full of hard math problems!

Edelstein, Guillén, Solomon, Ben-Chen (2023)

geodesic distances

Crane, Weischedel, Wardetzky (2013)

shape co

Abdelreheem, Eldesokey, Ovsjanikov, Wonk

optimal transport

Solomon, Goes, Peyre, Cuturi, Butscher, Nguyen, Du, Guibas (2015)

REFORMULATED AS AN OPTIMIZATION PROBLEM

$$\min \text{ or } \max\ f(x)$$

$$\text{subject to } x \in \Pi$$

THE FUNDAMENTALS

REFORMULATED AS AN OPTIMIZATION PROBLEM

$$\text{min or max } f(x)$$
$$\text{subject to } x \in \Pi$$

THE FUNDAMENTALS

find the x that makes the **objective function** $f(x)$ as small or big as possible

REFORMULATED AS AN OPTIMIZATION PROBLEM

$$\text{min or max } f(x)$$

$$\text{subject to } x \in \Pi$$

THE FUNDAMENTALS

find the x that makes the **objective function** $f(x)$ as small or big as possible

constraints define the set of all x that we are allowed to consider

REFORMULATED AS AN OPTIMIZATION PROBLEM

$$\text{min or max } f(x)$$

$$\text{subject to } x \in \Pi$$

THE FUNDAMENTALS

find the x that makes the **objective function** $f(x)$ as small or big as possible

constraints define the set of all x that we are allowed to consider

the set of all x that satisfy the constraints is called **feasible**

REFORMULATED AS AN OPTIMIZATION PROBLEM

$$\text{min or max } f(x)$$

$$\text{subject to } x \in \Pi$$

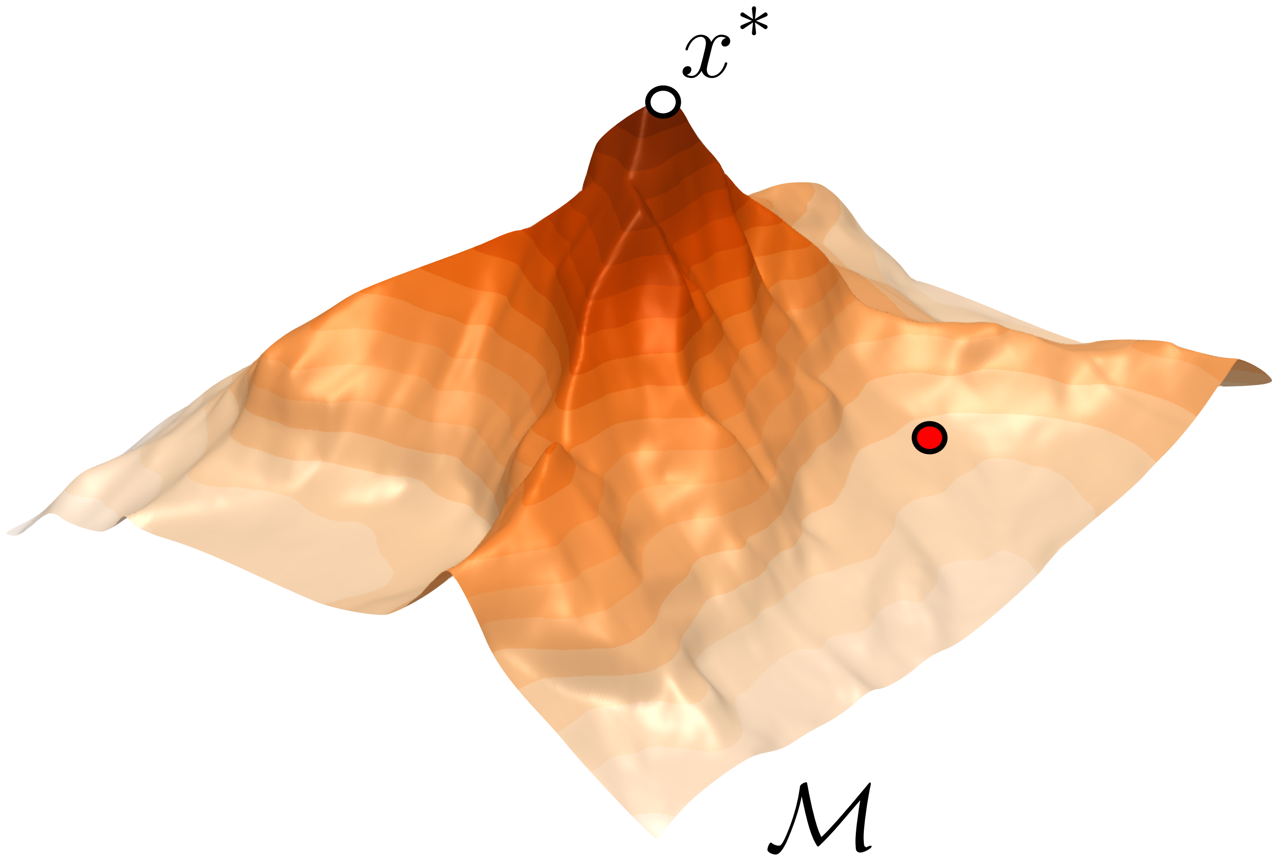
THE FUNDAMENTALS

$$x^* = \arg\max_{x \in \mathcal{M}} f(x)$$

From calculus...

$$\nabla f(x) = \mathbf{0}$$

Identifies the critical points



Original mesh by lastloginname via Thingiverse.
Modified version from odedstein-meshes.

THE FUNDAMENTALS

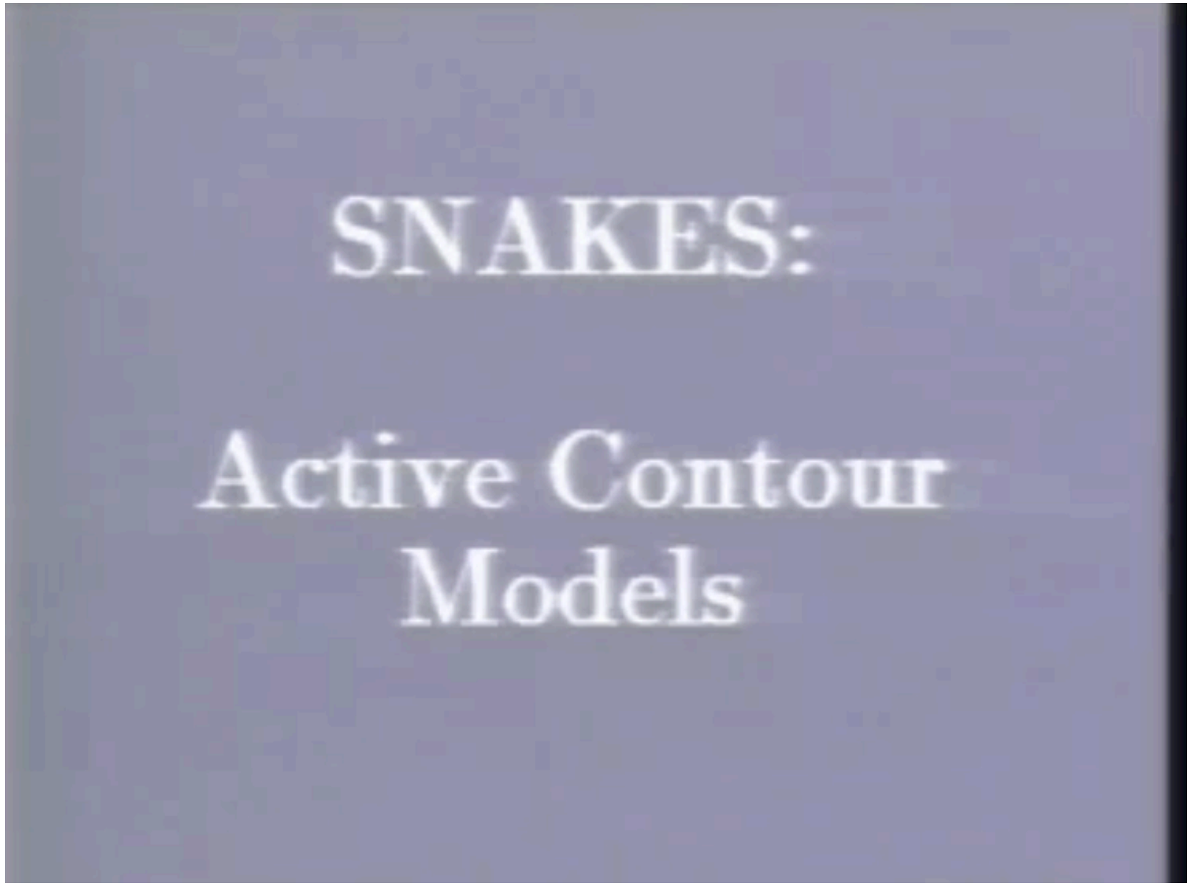
Snakes: Active Contour Models

Kass, Witkin and Terzopoulos (1988)

A snake is an **energy-minimizing spline** guided by external constraint forces and influenced by image forces that pull it toward features such as lines and edges.

$$v(s) = (x(s), y(s)), \ s \in [0, 1]$$

$$\text{minimize}\, \mathcal{E}_{\text{snake}}(v)$$



THE FUNDAMENTALS

Snakes: Active Contour Models

Kass, Witkin and Terzopoulos (1988)

A snake is an **energy-minimizing spline** guided by external constraint forces and influenced by image forces that pull it toward features such as lines and edges.

$$v_{i+1} = v_i - \gamma \nabla \mathcal{E}_{\text{snake}}(v)$$

🔑 gradient descent

The result is **highly sensitive to initialization** because classical "snakes" models are **non-convex** energy minimization problems solved using gradient descent.

OUR ROADMAP

OPTIMIZATION

1. The Fundamentals
2. Convexity
3. Why Convexity?
4. Standard Convex Problems

CONVEX RELAXATION

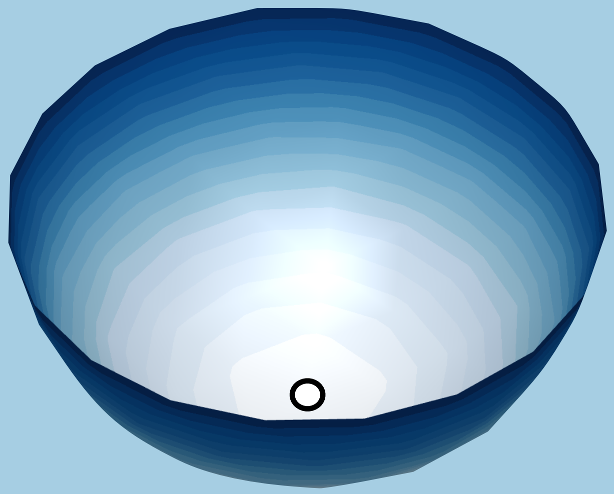
5. Viscosity Solutions
6. Mapping Problems
7. Optimal Transport
8. SOS Relaxations
9. Convex Substructures

SIMPLIFIED TAXONOMY OF CONVEX PROBLEMS

CONVEXITY

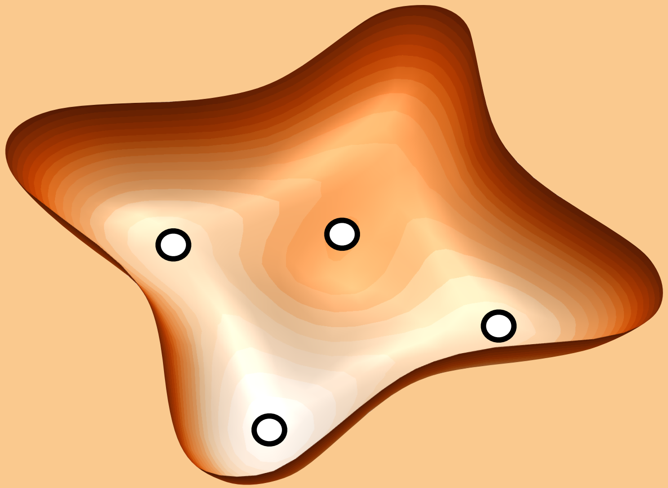
CONVEX
local optima are global optima

Convexity gives global optima



NON-CONVEX
multiple local optima

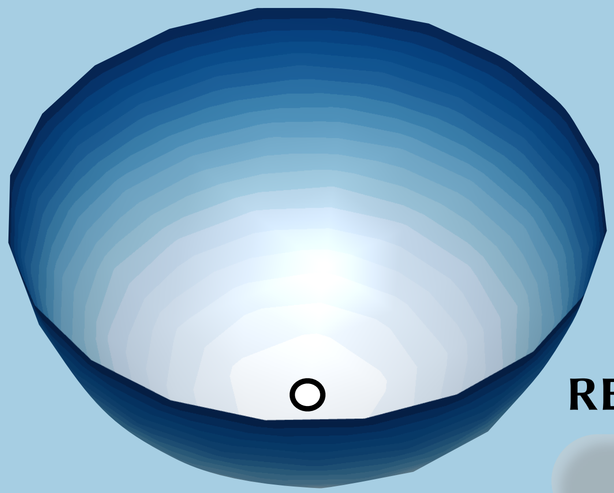
Often gets stuck on local optima or fails



CONVEXITY

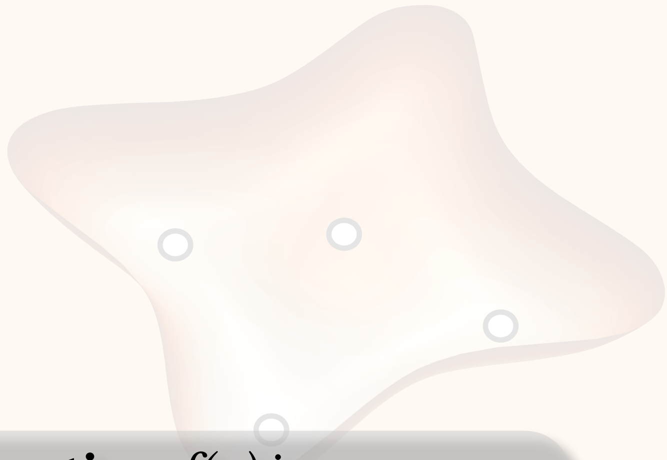
CONVEX
local optima are global optima

Convexity gives global optima



NON-CONVEX
multiple local optima

Often gets stuck on local optima or fails



REQUIREMENTS:

the **objective function** $f(x)$ is convex

constraint set is convex

CONVEXITY

CONVEX

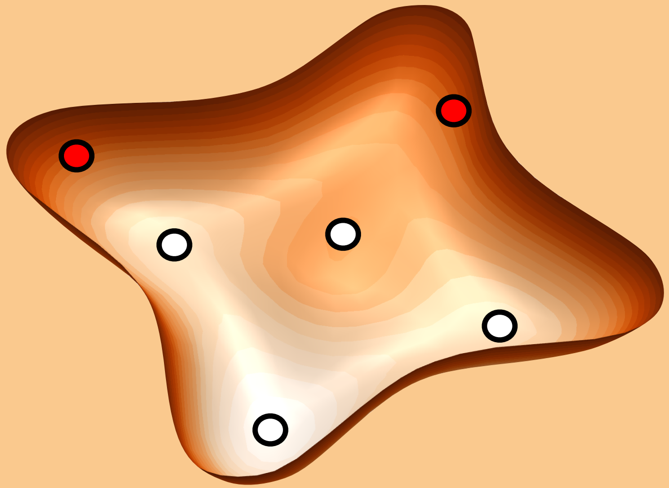
local optima are global optima

Convexity gives global optima

NON-CONVEX

multiple local optima

Often gets stuck on local optima or fails



THE ISSUE WITH NON-CONVEX PROBLEMS:

does not always recover the global optima

where we start matters!

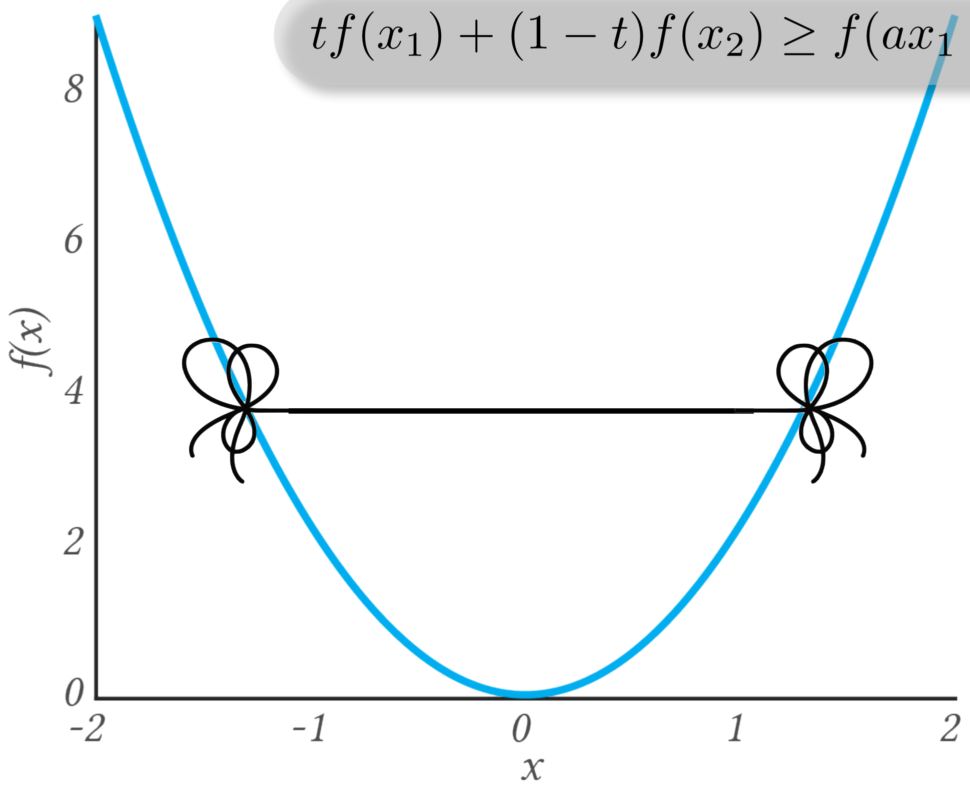
CONVEXITY

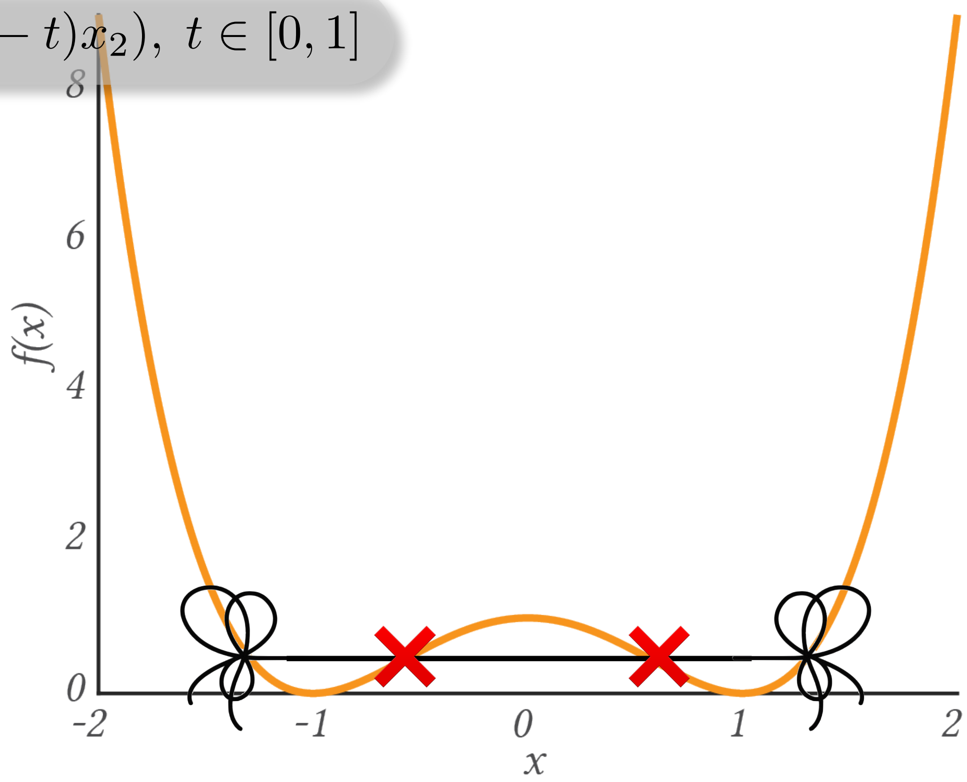
CONVEX FUNCTION

Convex functions lie below their **bowstrings**

mathematically

$$tf(x_1) + (1-t)f(x_2) \geq f(ax_1 + (1-t)x_2),\ t \in [0,1]$$





CONVEXITY

some standard convex functions...

linear functions

$$f(x) = c^\top x + d$$

norms

$$f(x) = \|x\|_p$$

exponential

$$f(x) = e^{ax}$$

certain powers

$$f(x) = x^a$$
$$x > 0,\ a \notin (0,1)$$

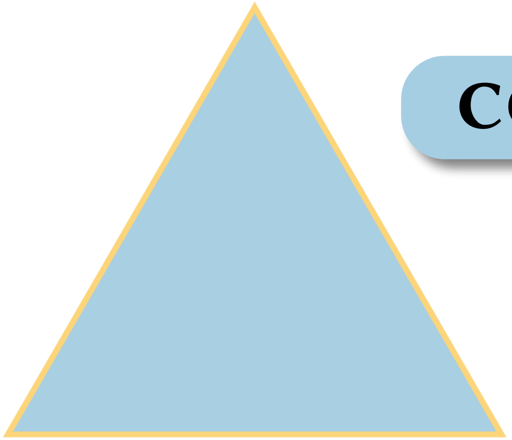
CONVEXITY

CONVEX SETS

Convex sets **don't** lie below a rubber band

mathematically

$$tx_1 + (1-t)x_2 \in \Pi, \ \forall x_1, x_2 \in \Pi, t \in [0,1]$$

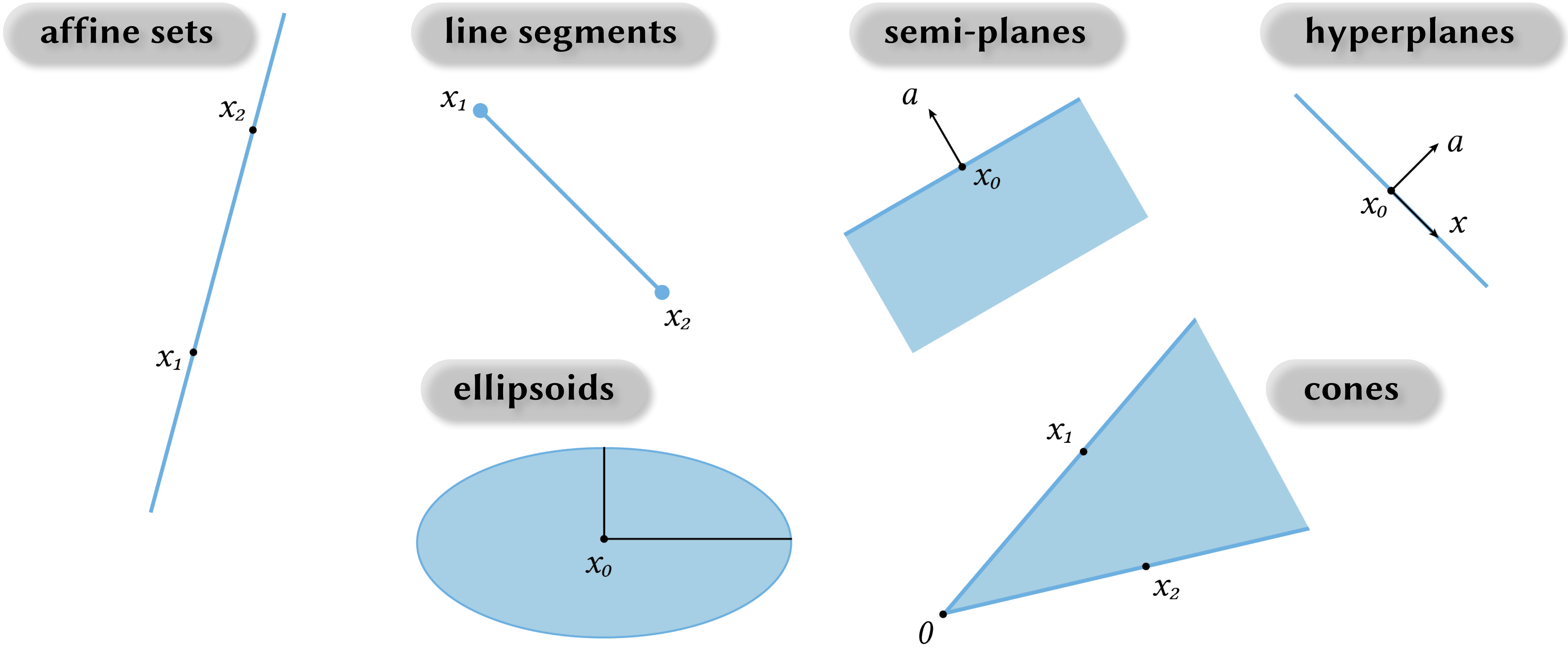


CONVEX

NON-CONVEX!

CONVEXITY

some standard convex sets…



OUR ROADMAP

OPTIMIZATION

1. The Fundamentals
2. Convexity
3. Why Convexity?
4. Standard Convex Problems

CONVEX RELAXATION

5. Viscosity Solutions
6. Mapping Problems
7. Optimal Transport
8. SOS Relaxations
9. Convex Substructures

SIMPLIFIED TAXONOMY OF CONVEX

WHY CONVEXITY?

we can solve convex problems with efficiency and robustness

Convexity **gives global optima** and theoretical guarantees

WHY CONVEXITY?

we can solve convex problems with efficiency and robustness

Convexity **gives global optima** and theoretical guarantees

Algorithms are plentiful and "self-sufficient"

WHY CONVEXITY?

we can solve convex problems with efficiency and robustness

Convexity **gives global optima** and theoretical guarantees

Algorithms are plentiful and "self-sufficient"

Modern convex optimization software lets you **solve large, complex problems quickly**

🛠 **help you prototype quicker!**

mosek

GUROBI OPTIMIZATION

YALMIP

SCS SPLITTING CONIC SOLVER

WHY CONVEXITY?

we can solve convex problems with efficiency and robustness

Convexity **gives global optima** and theoretical guarantees

Algorithms are plentiful and "self-sufficient"

Modern convex optimization software lets you **solve large, complex problems quickly**

Convex optimization software **helps you debug**

🛠 **help you prototype quicker!**

mosek

GUROBI OPTIMIZATION

YALMIP

SCS SPLITTING CONIC SOLVER

WHY CONVEXITY?

```
% From Boyd & Vandenberghe, "Convex Optimization"
% Joëlle Skaf - 09/26/05
%
% Solves the following QP with inequality constraints:
%           minimize    1/2x'*P*x + q'*x + r
%               s.t.    -1 <= x_i <= 1       for i = 1,2,3
% Also shows that the given x_star is indeed optimal

% Generate data
P = [13 12 -2; 12 17 6; -2 6 12];
q = [-22; -14.5; 13];
r = 1;
n = 3;
x_star = [1;1/2;-1];

% Construct and solve the model
fprintf(1,'Computing the optimal solution ...');
cvx_begin
    variable x(n)
    minimize ( (1/2)*quad_form(x,P) + q'*x + r)
    x >= -1;
    x <=  1;
cvx_end
fprintf(1,'Done! \n');

% Display results
disp('------------------------------------------------------------------------');
disp('The computed optimal solution is: ');
disp(x);
disp('The given optimal solution is: ');
disp(x_star);
```

🛠️ **help you prototype quicker!**

```
Interior-point solution summary
  Problem status  : PRIMAL_AND_DUAL_FEASIBLE
  Solution status : OPTIMAL
  Primal.  obj: 3.9124999974e+01    nrm: 3e+01    Viol.  con: 3e-08    var: 0e+00
  Dual.    obj: 3.9124999978e+01    nrm: 2e+00    Viol.  con: 0e+00    var: 3e-09
Optimizer summary
  Optimizer                 -                        time: 0.02
    Interior-point          - iterations : 10        time: 0.01
      Basis identification  -                        time: 0.00
        Primal              - iterations : 0         time: 0.00
        Dual                - iterations : 0         time: 0.00
        Clean primal        - iterations : 0         time: 0.00
        Clean dual          - iterations : 0         time: 0.00
    Simplex                 -                        time: 0.00
      Primal simplex        - iterations : 0         time: 0.00
      Dual simplex          - iterations : 0         time: 0.00
    Mixed integer           - relaxations: 0         time: 0.00

------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): -21.625

Done!
------------------------------------------------------------------------
The computed optimal solution is:
    1.0000
    0.5000
   -1.0000

The given optimal solution is:
    1.0000
    0.5000
   -1.0000
```

WHY NOT CONVEXITY?

WHY NOT CONVEXITY?

NON-CONVEX PROBLEM

What if you are given a bad hand?

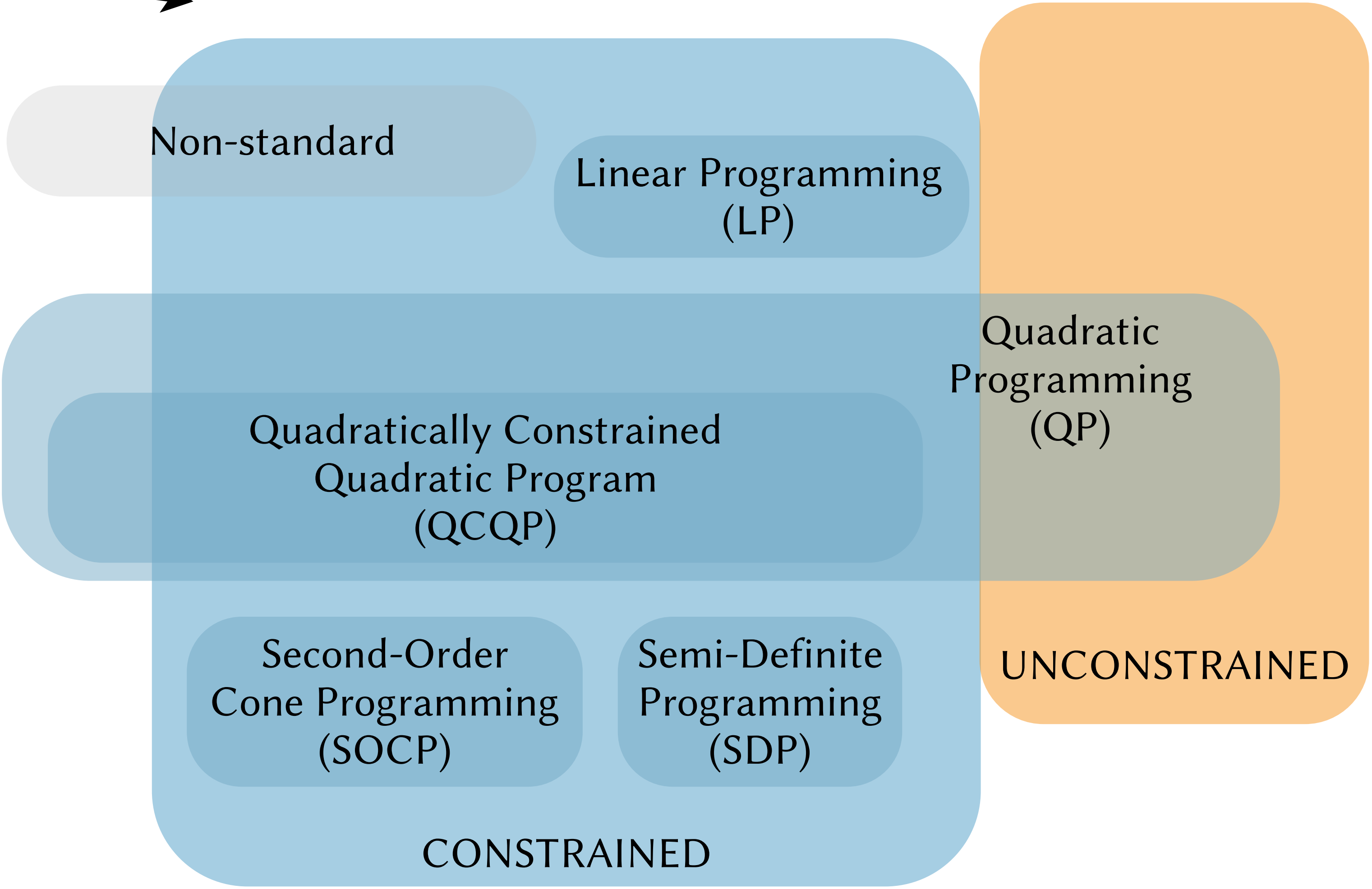
WHY NOT CONVEXITY?

Convex Relaxation

NON-CONVEX PROBLEM

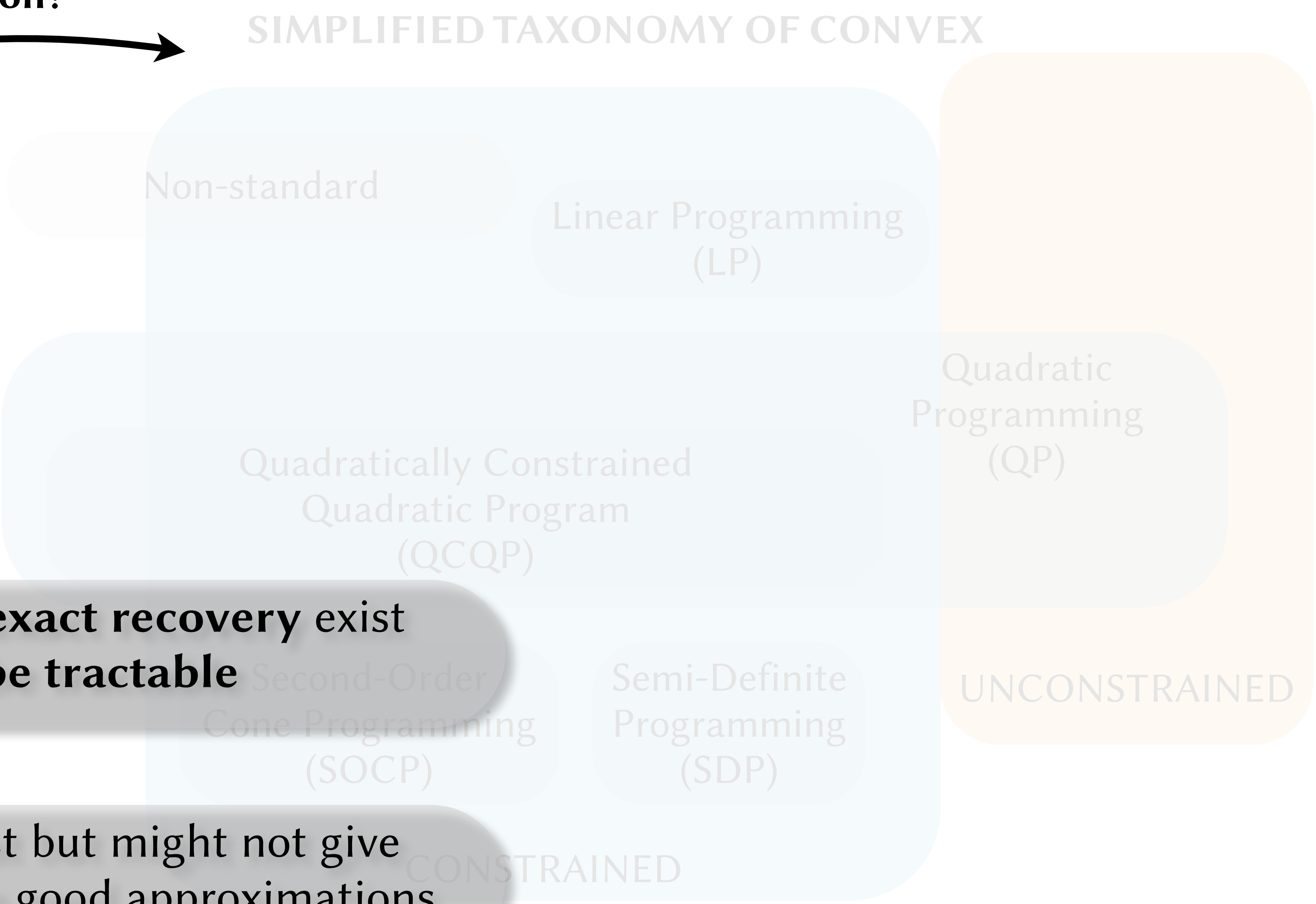
What if you are given a bad hand?

SIMPLIFIED TAXONOMY OF CONVEX



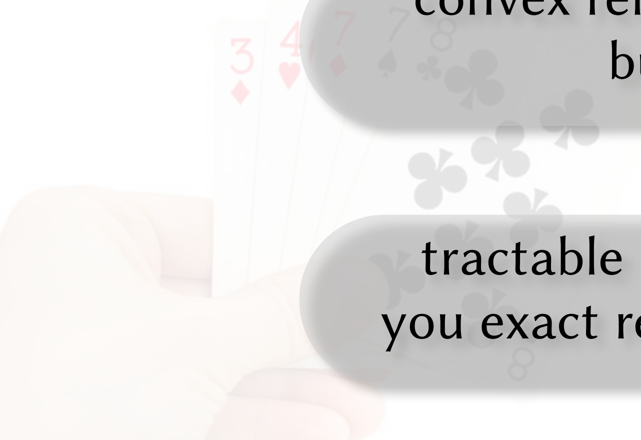
WHY **NOT** CONVEXITY?

Convex Relaxation?



NON-CONVEX PROBLEM

What if you are given a bad hand?



convex relaxations with **exact recovery** exist but **might not be tractable**

tractable relaxations exist but might not give you exact recovery or even good approximations

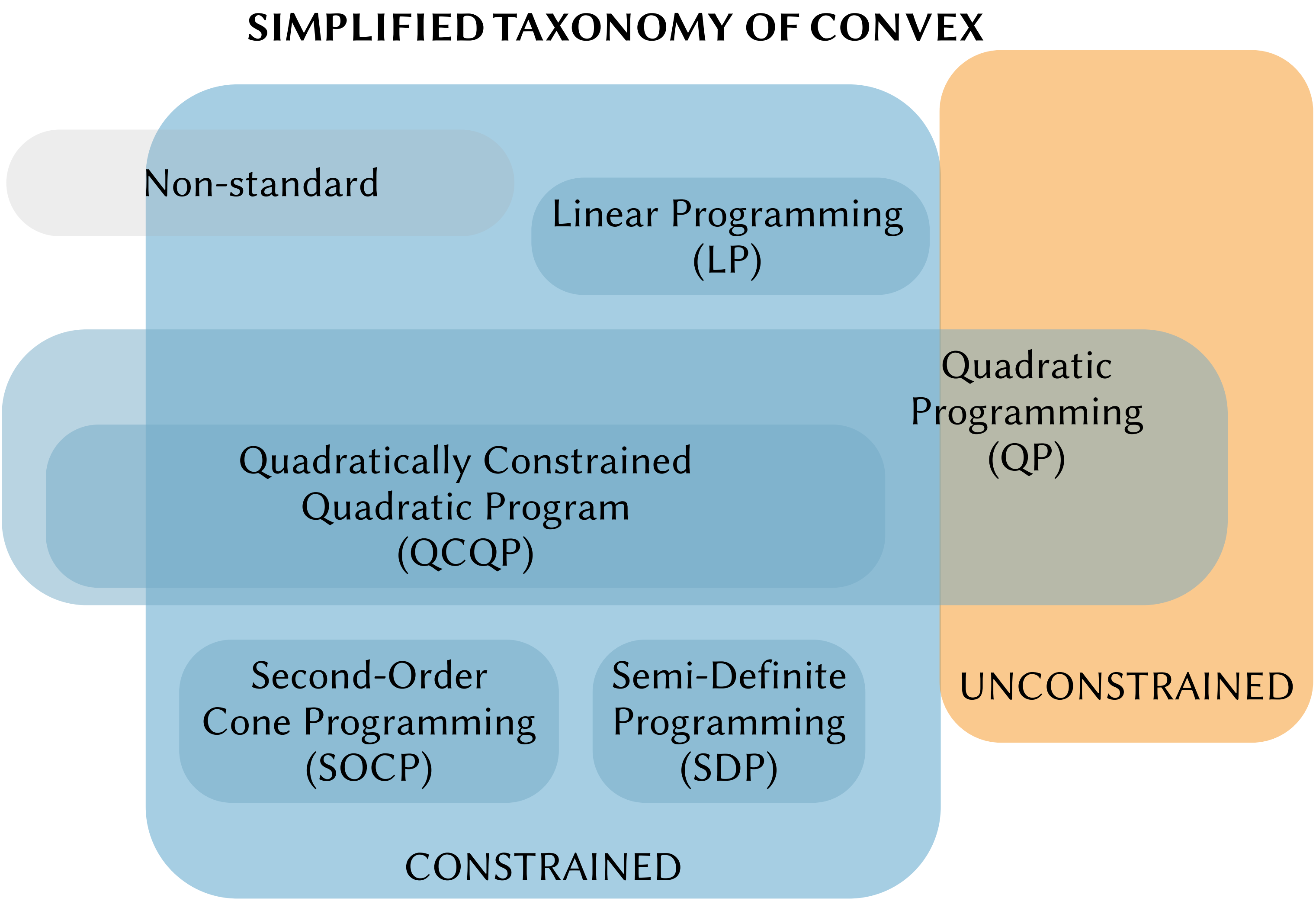
OUR ROADMAP

OPTIMIZATION

1. The Fundamentals
2. Convexity
3. Why Convexity?
4. Standard Convex Problems

CONVEX RELAXATION

5. Viscosity Solutions
6. Mapping Problems
7. Optimal Transport
8. SOS Relaxations
9. Convex Substructures



OUR ROADMAP

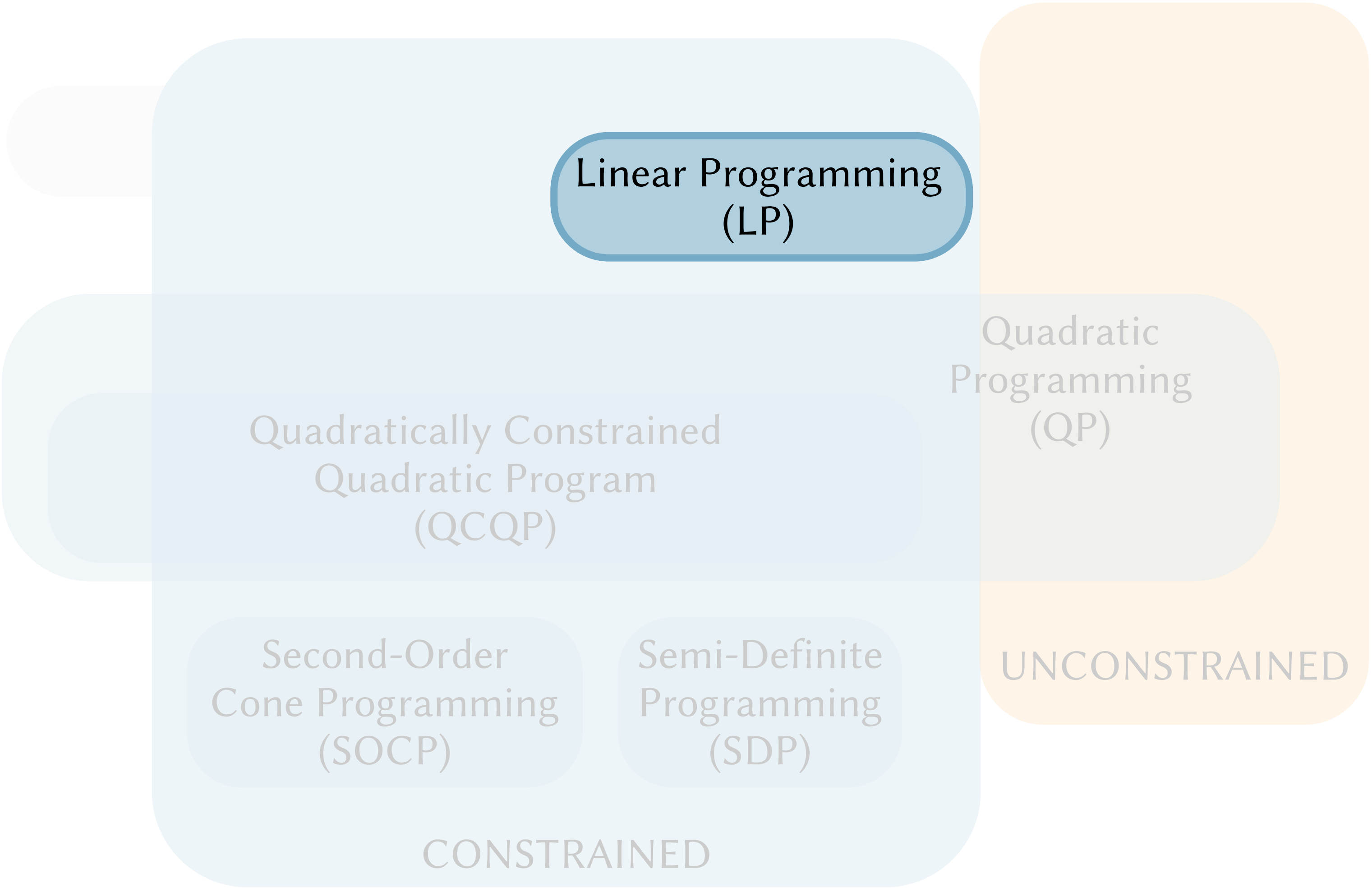
OPTIMIZATION

1. The Fundamentals
2. Convexity
3. Why Convexity?
4. Standard Convex Problems

CONVEX RELAXATION

5. Viscosity Solutions
6. Mapping Problems
7. Optimal Transport
8. SOS Relaxations
9. Convex Substructures

SIMPLIFIED TAXONOMY OF CONVEX



STANDARD CONVEX PROBLEMS

LINEAR PROGRAMMING (LP)

$$\text{minimize} \quad c^\top x + d$$
$$\text{subject to} \quad Ax \leq b$$

find the variable x that...

minimizes the **linear objective function**

subject to the **linear inequality constraints**

feasible set is a **convex polytope!**

STANDARD CONVEX PROBLEMS

LINEAR PROGRAMMING (LP)

$$\text{minimize} \quad c^\top x + d$$
$$\text{subject to} \quad Ax \leq b$$

find the variable x that...

minimizes the **linear objective function**

subject to the **linear inequality constraints**

feasible set is a **convex polytope!**

intersection of **semi-planes**

$$a_{11}x_{11} + a_{12}x_{12} + \cdots + a_{1n}x_{1n} \leq b_1$$
$$\vdots$$
$$a_{m1}x_{m1} + a_{m2}x_{m2} + \cdots + a_{mn}x_{mn} \leq b_m$$

STANDARD CONVEX PROBLEMS

Optimal transport

Earth Mover's Distance

$$\mathcal{W}(\mu_0, \mu_1) := \inf_\pi \int\int d(x,y)\mathrm{d}\pi(x,y)$$

$$\text{s. t. } \pi \in \Pi(\mu_0, \mu_1)$$

seeking a joint probability distribution

👍 convex linear programming

STANDARD CONVEX PROBLEMS

Optimal transport

Earth Mover's Distance

$$\mathcal{W}(\mu_0, \mu_1) := \inf_\pi \int\int d(x,y)\mathrm{d}\pi(x,y)$$
$$\text{s.t. } \pi \in \Pi(\mu_0, \mu_1)$$

seeking a joint probability distribution

👍 convex linear programming

👎 scales quadratically with the number of variables

STANDARD CONVEX PROBLEMS

Optimal transport

Earth Mover's Distance

$$\mathcal{W}(\mu_0, \mu_1) := \inf_\pi \int\int d(x,y)\mathrm{d}\pi(x,y)$$
$$\text{s.t.} \quad \pi \in \Pi(\mu_0, \mu_1)$$

seeking a joint probability distribution

👍 convex linear programming

👎 scales quadratically with the number of variables

💡 lesson: **convex does not always mean tractable!**

Earth Mover's Distances on Discrete Surfaces

Solomon, Rustamov, Guibas, Butscher (2014)

$$\mathcal{W}(\mu_0, \mu_1) = \inf_J \int \|J(x)\|\mathrm{d}x$$
$$\text{s.t.} \quad \nabla \cdot J(x) = \rho_1(x) - \rho_0(x)$$
$$J(x) \cdot n(x) = 0 \ \forall x \in \partial M$$

OUR ROADMAP

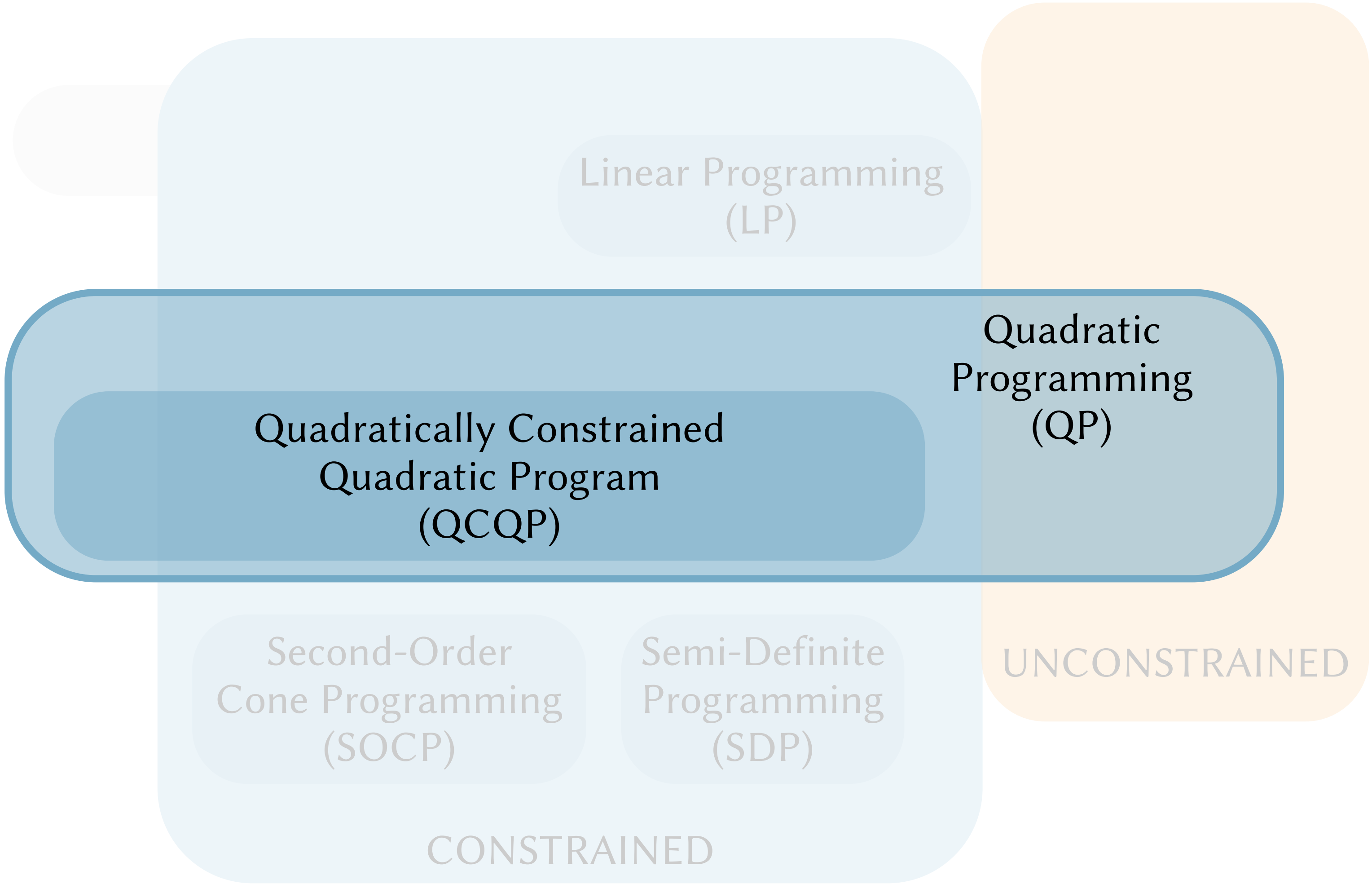
OPTIMIZATION

1. The Fundamentals
2. Convexity
3. Why Convexity?
4. Standard Convex Problems

CONVEX RELAXATION

5. Viscosity Solutions
6. Mapping Problems
7. Optimal Transport
8. SOS Relaxations
9. Convex Substructures

SIMPLIFIED TAXONOMY OF CONVEX



STANDARD CONVEX PROBLEMS

QUADRATIC PROGRAMMING (QP)

$$\text{minimize} \quad \frac{1}{2}x^\top Qx + c^\top x + d$$

$$\text{subject to} \quad Ax \leq b$$

find the variable x that...

minimizes the **quadratic objective function**

subject to the **linear inequality constraints**

$Q \in \mathbb{S}^n_+$ so the objective is **convex quadratic**

Positive Semi-Definite (PSD)
A symmetric matrix M is PSD if
$$v^\top Mv \geq 0$$
for every vector v.

STANDARD CONVEX PROBLEMS

Shape deformation

Bounded Biharmonic Weights for Real-Time Deformation

Jacobson, Baran, Popović, Sorkine (2011)

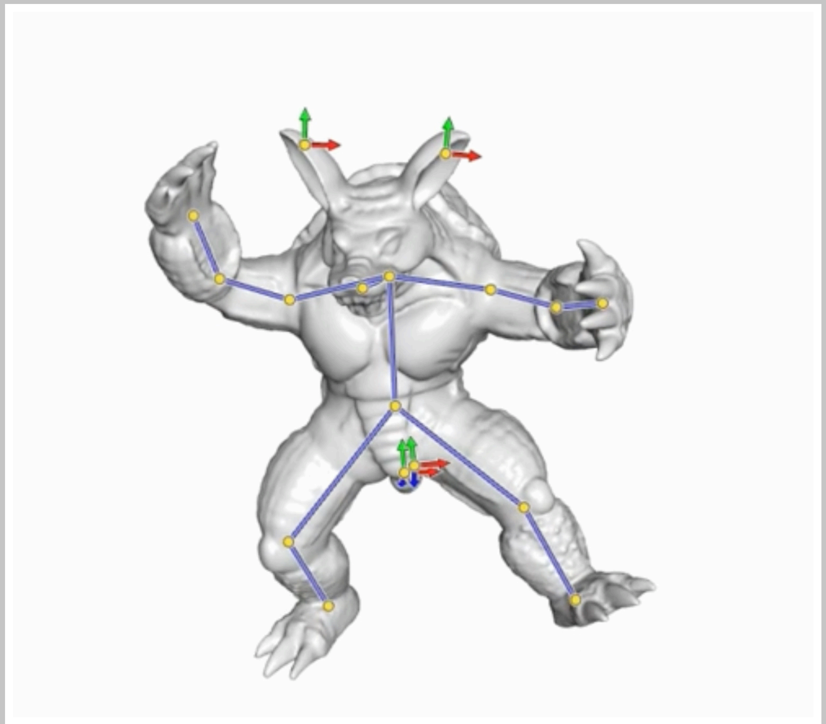
$$\underset{w_j,\ j=1,\ldots,m}{\arg\min} \sum_{j=1}^{m} \frac{1}{2} \int_{\Omega} \|\Delta w_j\|^2 \mathrm{d}V$$

$$\text{s.t.}\quad w_j|_{H_k} = \delta_{jk}$$

$$w_j|_F \text{ is linear}$$

$$\sum_{j=1}^{m} w_j(\mathbf{p}) = 1$$

$$0 \le w_j(\mathbf{p}) \le 1,\ j = 1, \ldots, m$$



OUR ROADMAP

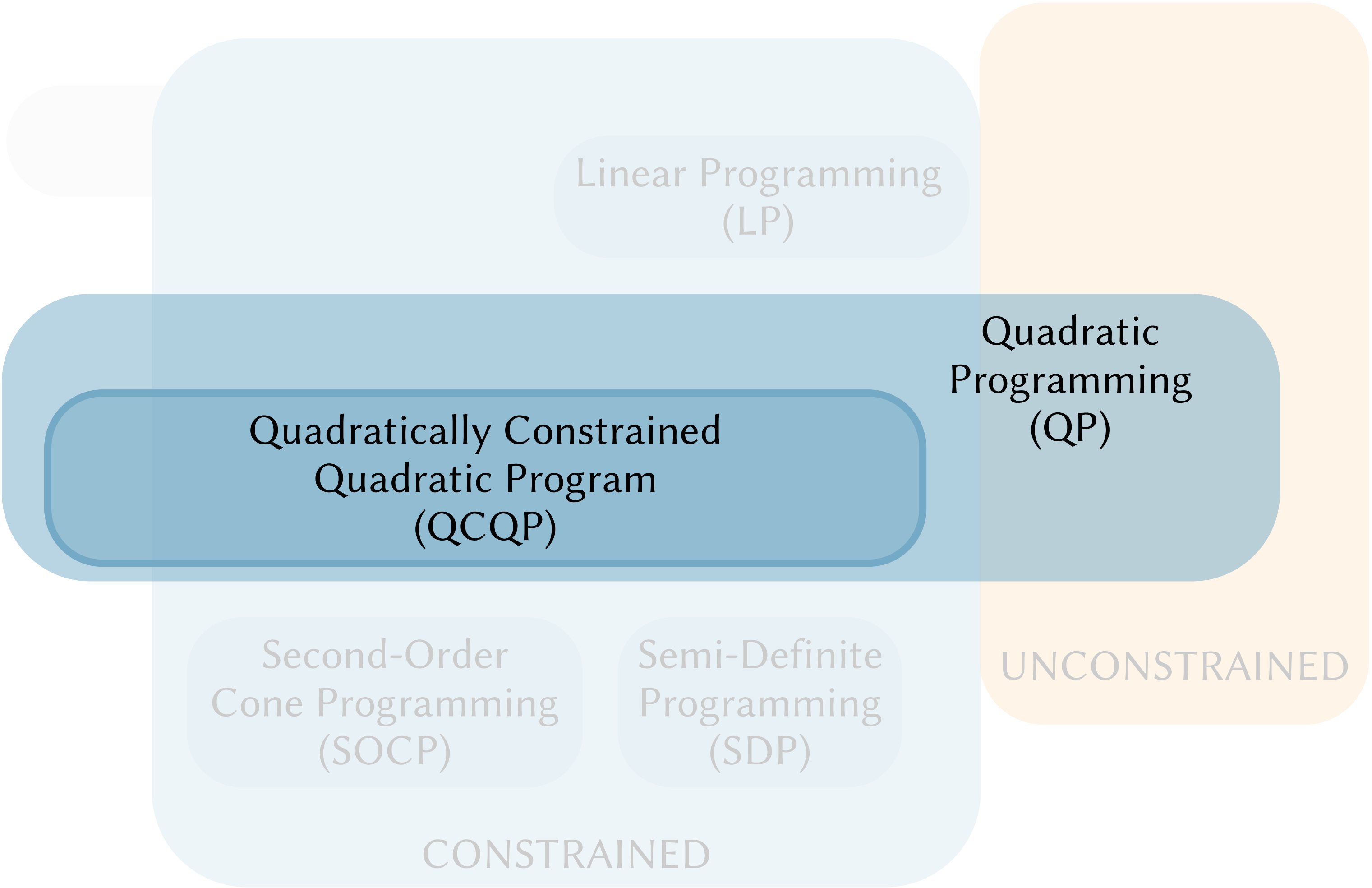
OPTIMIZATION

1. The Fundamentals
2. Convexity
3. Why Convexity?
4. Standard Convex Problems

CONVEX RELAXATION

5. Viscosity Solutions
6. Mapping Problems
7. Optimal Transport
8. SOS Relaxations
9. Convex Substructures

SIMPLIFIED TAXONOMY OF CONVEX



STANDARD CONVEX PROBLEMS

QUADRATICALLY CONSTRAINED QUADRATIC PROGRAMMING (QCQP)

$$\text{minimize } \; 1/2\, x^\top Q x + c^\top x + d$$

$$\text{subject to } \; 1/2\, x^\top P_i x + q_i^\top x + r_i \leq 0$$

$Q, P \in \mathbb{S}^n_+$ so the objective and constraints are **convex quadratic**

find the variable x that...

minimizes the **quadratic objective function**

subject to the **quadratic inequality constraints**

STANDARD CONVEX PROBLEMS

QUADRATICALLY CONSTRAINED QUADRATIC PROGRAMMING (QCQP)

$$\text{minimize}\;\; 1/2\, x^\top Q x + c^\top x + d$$

$$\text{subject to}\;\; 1/2\, x^\top P_i x + q_i^\top x + r_i \leq 0$$

$Q, P \in \mathbb{S}^n_+$ so the objective and constraints are **convex quadratic**

find the variable x that...

minimizes the **quadratic objective function**

subject to the **quadratic inequality constraints**

feasible set is the **intersection of ellipsoids!**

WHAT C DONE?

Point-to-poi

Procrustes matching

$$\text{minimize } \|RP - QX\|_F^2$$
$$\text{s.t. } R^\top R = I, X \text{ a permutation}$$

STANDA NVEX PROBLEMS

Point-to-poi

Procrustes matching

$$\begin{aligned}&\text{minimize}\ \|RP - QX\|_F^2\\ &\text{s.t.}\ \ R^\top R = I, X \text{ a permutation}\end{aligned}$$

Question: is this problem convex?

STANDA NVEX PROBLEMS

Point-to-poi

Procrustes matching

$$\begin{aligned} &\text{minimize } \|RP - QX\|_F^2 \\ &\text{s.t. } R^\top R = I, X \text{ a permutation} \end{aligned}$$

Question: is this problem convex? **NO**

STANDA NVEX PROBLEMS

Point-to-poi

Procrustes matching

$$\text{minimize } \|RP - QX\|_F^2$$
$$\text{s.t. } R^\top R = I, X \text{ a permutation}$$

Question: is this problem convex? **NO**

Point Registration via Efficient Convex Relaxation

Maron, Dym, Kezurer, Kovalsky, Lipman (2016)

We will see the relaxation later…

OUR ROADMAP

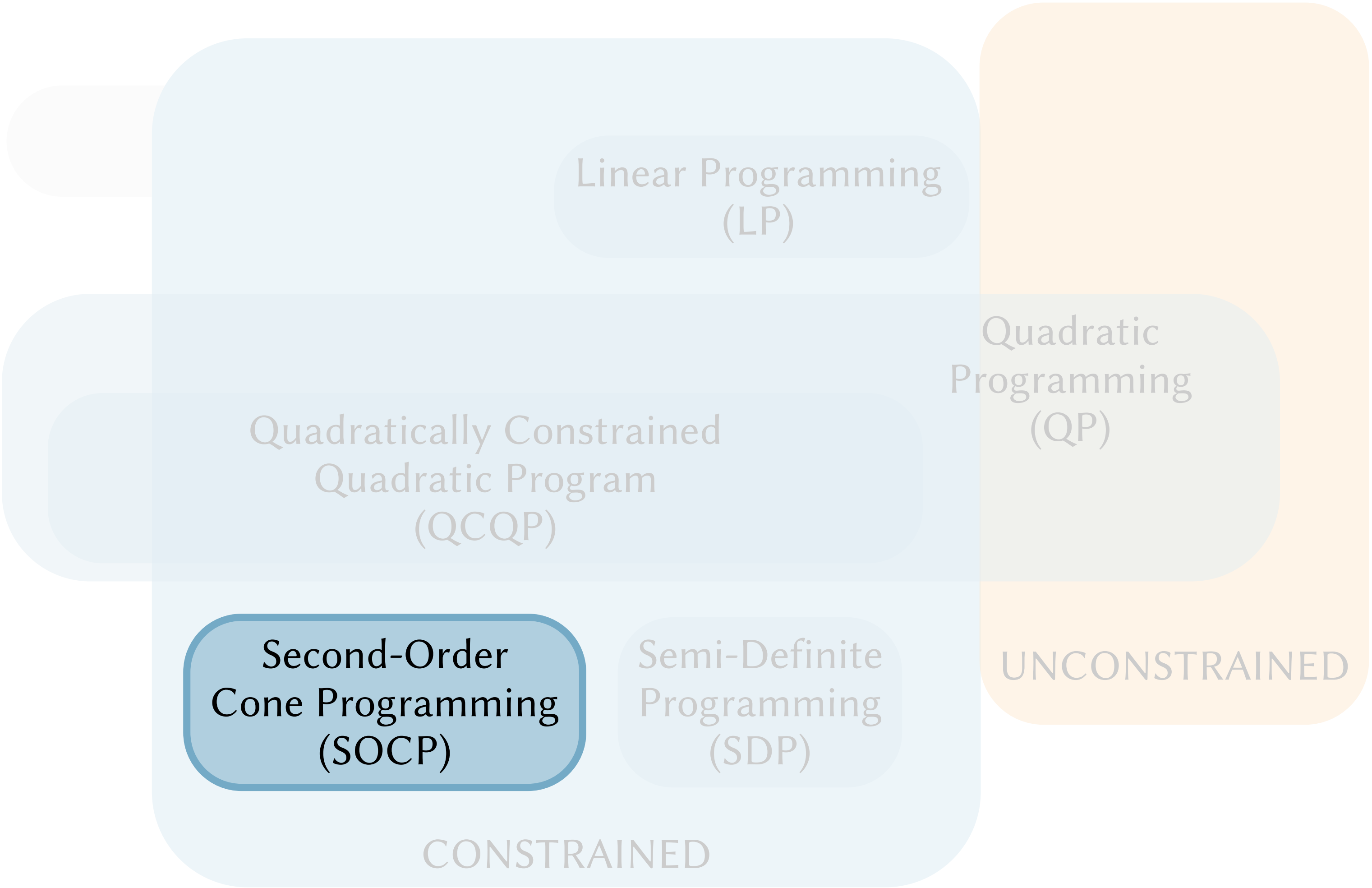
OPTIMIZATION

1. The Fundamentals
2. Convexity
3. Why Convexity?
4. Standard Convex Problems

CONVEX RELAXATION

5. Viscosity Solutions
6. Mapping Problems
7. Optimal Transport
8. SOS Relaxations
9. Convex Substructures

SIMPLIFIED TAXONOMY OF CONVEX



STANDARD CONVEX PROBLEMS

SECOND-ORDER CONE PROGRAMMING (SOCP)

$$\begin{aligned} \text{minimize} \quad & f^\top x \\ \text{subject to} \quad & \|A_i x + b_i\|_2 \le c_i^\top x + d_i \\ & Fx = q \end{aligned}$$

find the variable x that...

minimizes the **linear objective function**

subject to the **cone and linear constraints**

inequality constraints are **second-order cones**

STANDARD CONVEX PROBLEMS

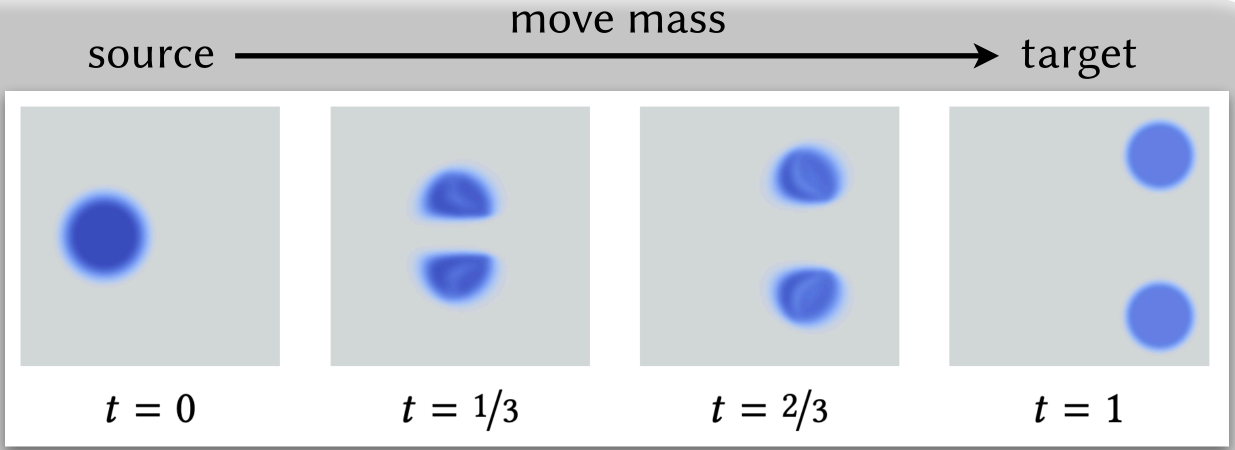
Optimal transport

Dynamical Optimal Transport on Discrete Surfaces

Lavenant, Claici, Chien, Solomon (2018)

$$\mathcal{W}(\mu_0, \mu_1) = \max_\phi \int \phi_1 \mathrm{d}\mu_1 - \int \phi_0 \mathrm{d}\mu_0$$

$$\text{s.t. } \partial_t \phi + \frac{1}{2}\|\nabla \phi\|^2 \leq 0$$



STANDARD CONVEX PROBLEMS

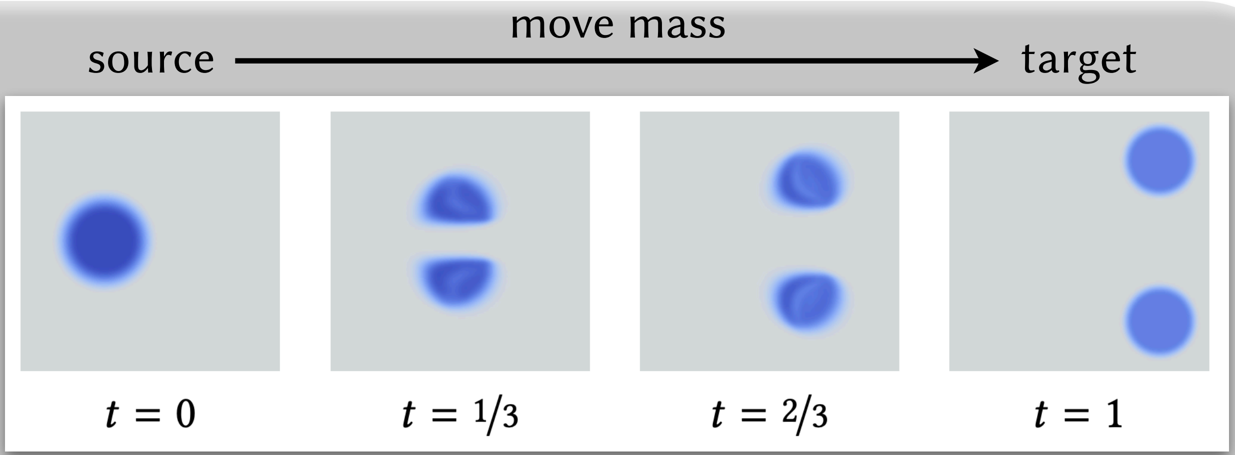
Optimal transport

Dynamical Optimal Transport on Discrete Surfaces

Lavenant, Claici, Chien, Solomon (2018)

$$\mathcal{W}(\mu_0, \mu_1) = \max_\phi \int \phi_1 \mathrm{d}\mu_1 - \int \phi_0 \mathrm{d}\mu_0$$

$$\text{s.t. } \partial_t \phi + \frac{1}{2}\|\nabla \phi\|^2 \leq 0$$



quadratic constraint?

Quadratic Inequality Constraints

$$x^\top Q x + c^\top x + d \leq 0$$

Cone constraints

$$\left\| Q^{1/2} x + \frac{1}{2} Q^{-1/2} c \right\|_2 \leq \left(\frac{1}{4} c^\top Q^{-1} c - d \right)^{1/2}$$

STANDARD CONVEX PROBLEMS

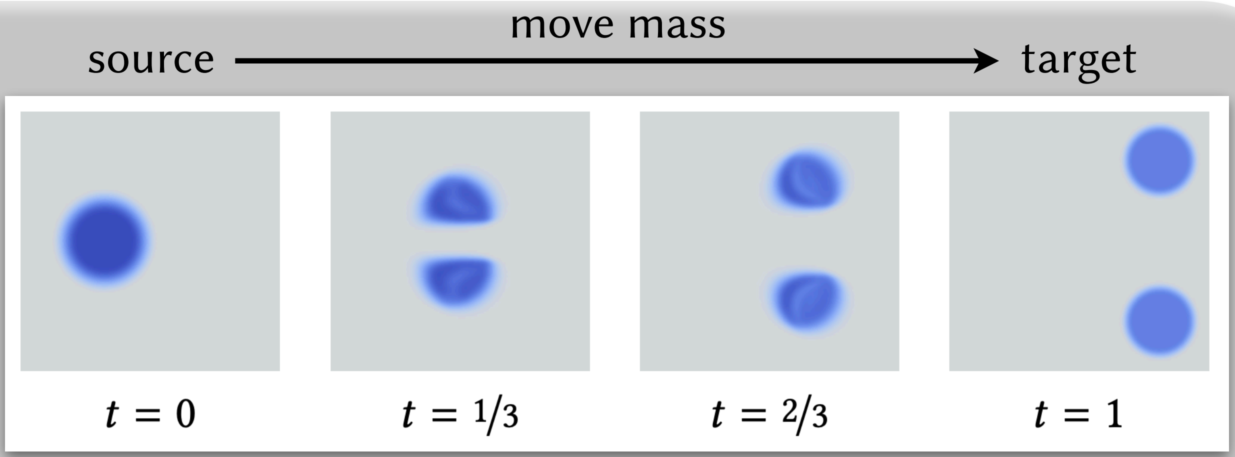
Optimal transport

Dynamical Optimal Transport on Discrete Surfaces

Lavenant, Claici, Chien, Solomon (2018)

$$\mathcal{W}(\mu_0, \mu_1) = \max_\phi \int \phi_1 \mathrm{d}\mu_1 - \int \phi_0 \mathrm{d}\mu_0$$

$$\mathrm{s.t.}\ \partial_t \phi + \frac{1}{2}\|\nabla \phi\|^2 \leq 0$$



quadratic constraint?

We will see it again later...

Quadratic Inequality Constraints

$$x^\top Q x + c^\top x + d \leq 0$$

Cone constraints

$$\left\| Q^{1/2} x + \frac{1}{2} Q^{-1/2} c \right\|_2 \leq \left(\frac{1}{4} c^\top Q^{-1} c - d \right)^{1/2}$$

OUR ROADMAP

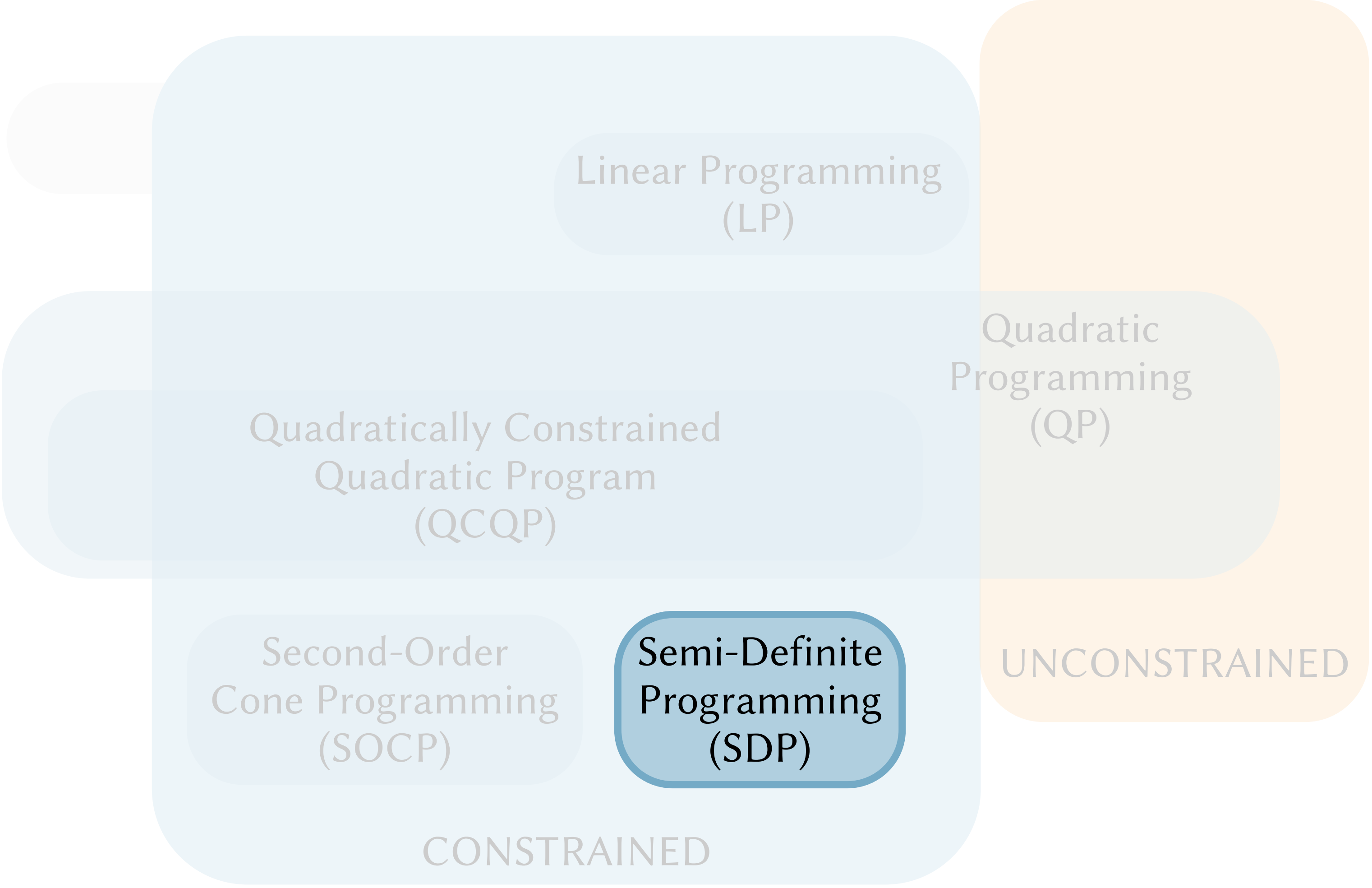
OPTIMIZATION

1. The Fundamentals
2. Convexity
3. Why Convexity?
4. Standard Convex Problems

CONVEX RELAXATION

5. Viscosity Solutions
6. Mapping Problems
7. Optimal Transport
8. SOS Relaxations
9. Convex Substructures

SIMPLIFIED TAXONOMY OF CONVEX



STANDARD CONVEX PROBLEMS

SEMIDEFINITE PROGRAMMING (SDP)

$$\text{minimize} \;\; \text{trace}(C^\top X)$$
$$\text{subject to} \;\; \text{trace}(A_i^\top X) \leq b_i$$
$$X \succeq 0$$

constraint is a **linear matrix inequality (LMI)**

find the variable *x* that...

minimizes the **linear objective function**

subject to the **LMI and PSD constraints**

STANDARD CONVEX PROBLEMS

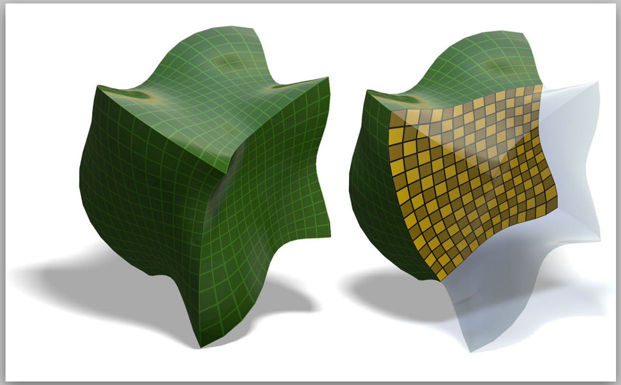
Volumetric mesh deformations

Controlling Singular Values with Semidefinite Programming
Kovalsky, Aigerman, Basri, Lipman (2014)

$$\min_{A \in \mathbb{R}^{n \times n}} ||A - B||_F$$

$$\text{s.t.} \begin{pmatrix} \Gamma I & A \\ A^T & \Gamma I \end{pmatrix} \succeq 0$$

$$\frac{A + A^T}{2} \succeq \gamma I$$



OUR ROADMAP

OPTIMIZATION

1. The Fundamentals
2. Convexity
3. Why Convexity?
4. Standard Convex Problems

CONVEX RELAXATION

5. Viscosity Solutions
6. Mapping Problems
7. Optimal Transport
8. SOS Relaxations
9. Convex Substructures

SIMPLIFIED TAXONOMY OF CONVEX

COFFEE BREAK

💭Questions?

OUR ROADMAP

OPTIMIZATION

1. The Fundamentals
2. Convexity
3. Why Convexity?
4. Standard Convex Problems

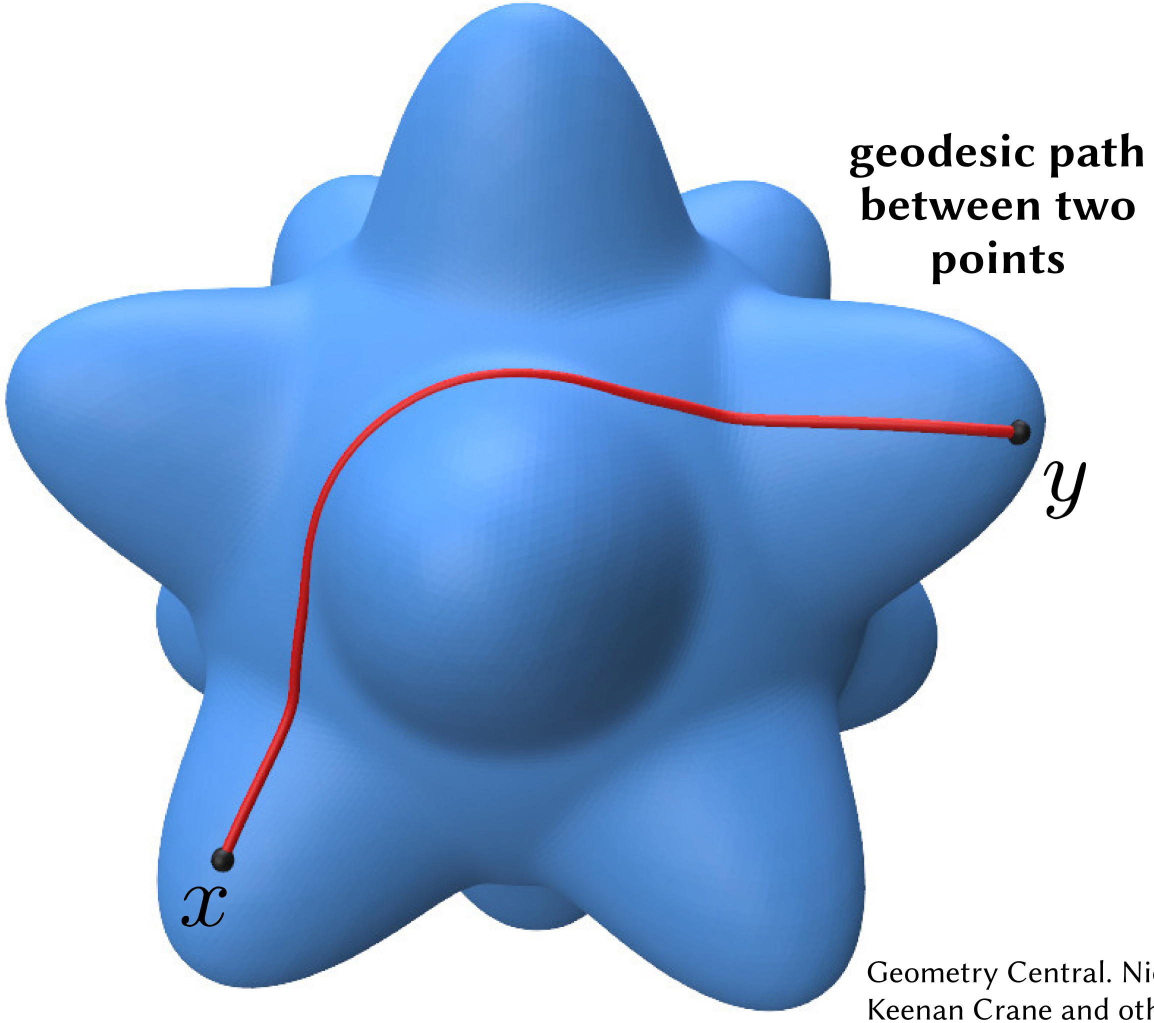
CONVEX RELAXATION

5. PDE Solutions
6. Mapping Problems
7. Optimal Transport
8. SOS Relaxations
9. Convex Substructures

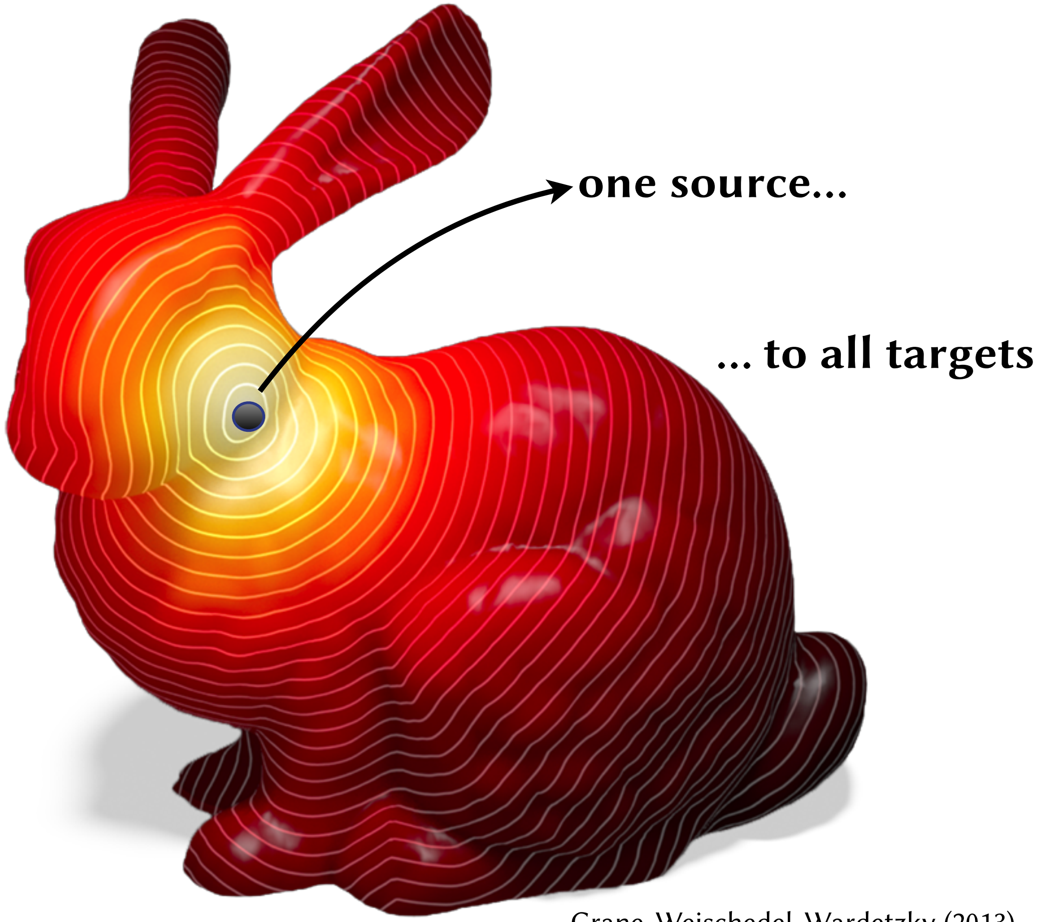
SIMPLIFIED TAXONOMY OF CONVEX

VISCOSITY SOLUTIONS

Computing geodesic distances



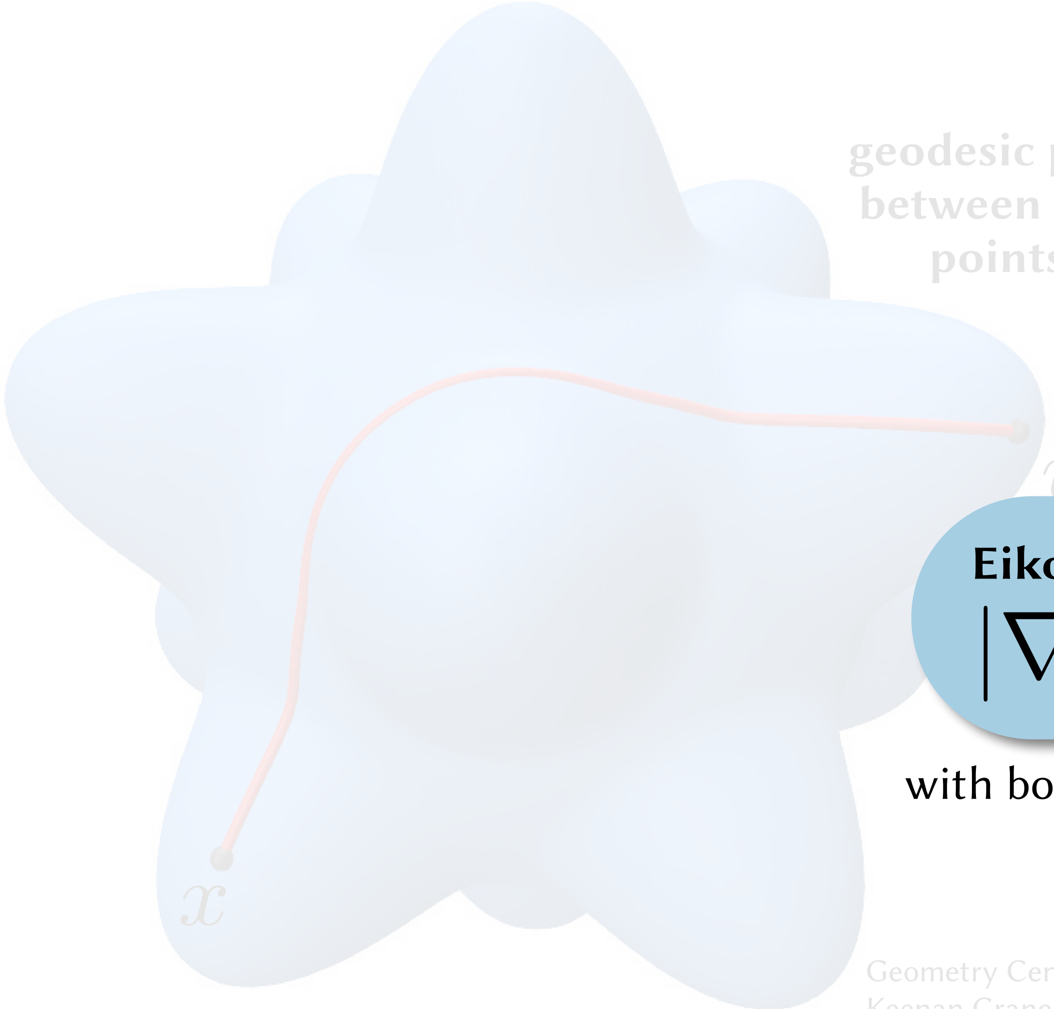
Geometry Central. Nicholas Sharp, Keenan Crane and others (2019)



Crane, Weischedel, Wardetzky (2013)

VISCOSITY SOLUTIONS

Computing geodesic distances



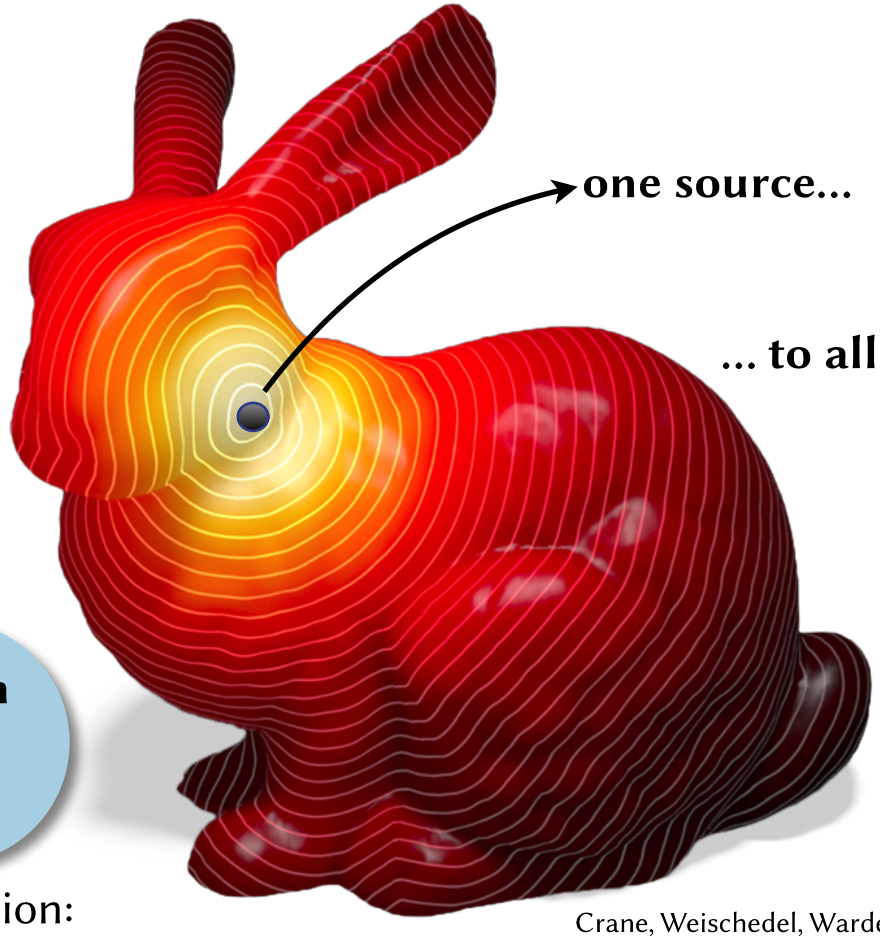
Geometry Central. Nicholas Sharp, Keenan Crane and others (2019)

Eikonal equation

$$|\nabla u| = 1$$

with boundary condition:

$$u|_{x_0} = 0$$



Crane, Weischedel, Wardetzky (2013)

VISCOSITY SOLUTIONS

Computing geodesic distances

An ADMM-based Scheme for Distance Function Approximation
Belyaev & Fayolle (2020)


Numerical Algorithms (2020) 84:983–996
https://doi.org/10.1007/s11075-019-00789-5
ORIGINAL PAPER

Check for updates

An ADMM-based scheme for distance function approximation

Alexander Belyaev[1] · Pierre-Alain Fayolle[2]

Received: 6 March 2019 / Accepted: 24 July 2019 / Published online: 19 August 2019
© The Author(s) 2019

Abstract
A novel variational problem for approximating the distance function (to a dom boundary) is proposed. It is shown that this problem can be efficiently solv ADMM. A review of several other variational and PDE-based methods tance function estimation is presented. Advantages of the proposed distance estimation method are demonstrated by numerical experiments. Applicat method to the problems of surface curvature estimation and computing of a binary image are shown.

Keywords Distance function · Variational methods · Distance trans Skeleton · Curvature

1 Introduction

Fast and accurate estimation of the distance to a surface (is important for a number of applications including redi level-set methods [17], wall distance models in turbul geneous material modeling in computational mech and meshing [31], FEM extensions [2, 16], robo

Alexander Belyaev
a.belyaev@hw.ac.uk

Pierre-Alain Fayolle
fayolle@u-aizu.ac.jp

1 School of Engineering & Physical Sc
2 Computer Graphics Laboratory, U

Numerical Algorithms (2020) 84:983–996 987

to the boundary function [5]. The main difference with the approach proposed here lies in the second step of minimizing (8) by ADMM, where a simple projection can now be used instead of numerically computing the root of a polynomial.

3 Numerical experiments: distance to a polygonal mesh using FEM and application to curvature computation

We consider first the problem of computing the distance function to $\partial\Omega$, a surface (curve in 2D), represented by a triangle mesh (a polygonal chain in 2D).

As described in Section 2, we use ADMM to minimize (7). This gives an iterative process, where the first step involves solving a Poisson problem of the form: $-\Delta\phi = f$. This Poisson problem is discretized and solved by the finite elements method. The computational domain bounded by $\partial\Omega$ is represented by a triangle mesh in 2D or a tetrahedral mesh in 3D. Linear basis functions are used at each node of the triangulation. The solution to the Poisson problem is obtained from numerically solving a linear system $A\Phi = b$, where the sparse matrix A, corresponding to a discretization of the Laplacian, is the same for each iteration of ADMM and can thus be prefactored (for example with the Cholesky decomposition).

3.1 Distance computation

Figure 1 illustrates the result obtained by our approach on a 2D polygonal domain with complex geometry. The left image visualizes the exact distance obtained by computing the minimum distance to any boundary segment. The middle image presents the distance computed by solving (7). The right image demonstrates how the relative residual error

$$\|\phi_{k+1} - \phi_k\|_2 / \|\phi_k\|_2$$

decreases with each iteration when solving (7) by ADMM. One can observe that (7) solved numerically by ADMM demonstrates a good convergence and is capable to deliver an accurate approximation of the distance function.

Fig. 1 Left: The exact distance computed as the min distance to the boundary segments. Middle: Distance to the boundary obtained from solving (7) by ADMM. Right: Relative residual error over a few ADMM iterations

Springer

VISCOSITY SOLUTIONS

Computing geodesic distances

An ADMM-based Scheme for Distance Function Approximation
Belyaev & Fayolle (2020)

💡 Key Insight

$$\max_\phi \int_\Omega \phi \mathrm{d}x \text{ s.t.} |\nabla \phi| \leq 1, \phi|_{\partial\Omega} = 0$$

VISCOSITY SOLUTIONS

Question: What if we want the distances to be regularized instead of approximate?

A Convex Optimization Framework for Regularized Geodesic Distances
Edelstein, Guillen, Solomon, Ben-Chen (2020)

Regularizer

$$\mathcal{E}(u) = \int_{\Omega} F(\nabla u(x), x)\mathrm{d}\,\mathrm{Vol}(x)$$

A Convex Optimization Framework for Regularized Geodesic Distances

Michal Edelstein
Technion - Israel Institute of Technology
Haifa, Israel
smichale@cs.technion.ac.il

Nestor Guillen
Texas State University
San Marcos, TX, USA
nestor@txstate.edu

Justin Solomon
Massachusetts Institute of Technology (MIT)
Cambridge, MA, USA
jsolomon@mit.edu

Mirela Ben-Chen
Technion - Israel Institute of Technology
Haifa, Israel
mirela@cs.technion.ac.il

Figure 1: Geodesic distances (a) may not have desired properties such as smoothness. We present a general framework for regularized geodesic distances. Shown here are three examples of regularizers: (b,c) smoothness, (d) alignment to a vector field, and (e) boundary invariance.

ABSTRACT

We propose a general convex optimization problem for computing *regularized* geodesic distances. We show that under mild conditions on the regularizer the problem is well posed. We propose three different regularizers and provide analytical solutions in special cases, as well as corresponding efficient optimization algorithms. Additionally, we show how to generalize the approach to the *all pairs* case by formulating the problem on the product manifold, which leads to *symmetric* distances. Our regularized distances compare favorably to existing methods, in terms of robustness and ease of calibration.

CCS CONCEPTS

• Computing methodologies → Shape analysis.

KEYWORDS

geodesic distance, convex optimization, triangle meshes

Permission to make digital or hard copies of part or all of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. Copyrights for third-party components of this work must be honored. For all other uses, contact the owner/author(s).
SIGGRAPH '23 Conference Proceedings, August 6–10, 2023, Los Angeles, CA, USA
© 2023 Copyright held by the owner/author(s).
ACM ISBN 979-8-4007-0159-7/23/08.
https://doi.org/10.1145/3588432.3591523

ACM Reference Format:
Michal Edelstein, Nestor Guillen, Justin Solomon, and Mirela Ben-Chen. 2023. A Convex Optimization Framework for Regularized Geodesic Distances. In *Special Interest Group on Computer Graphics and Interactive Techniques Conference Conference Proceedings (SIGGRAPH '23 Conference Proceedings), August 6–10, 2023, Los Angeles, CA, USA.* ACM, New York, NY, USA, 11 pages. https://doi.org/10.1145/3588432.3591523

1 INTRODUCTION

Distance computation is a central task in shape analysis. Distances are required for many downstream geometry processing applications, including shape correspondence, shape descriptors and remeshing. In many cases, however, exact geodesic distances are not required, and a *distance-like* function suffices. Moreover, it is often required to *regularize* the distance-like function to improve performance of a downstream application.

The geometry processing community has proposed myriad methods for computing geodesic distances [Crane et al. 2020], including some regularized distances [Crane et al. 2013; Solomon et al. 2014]. However, a unified framework, including a controlled and easily calibratable approach to regularization is still missing.

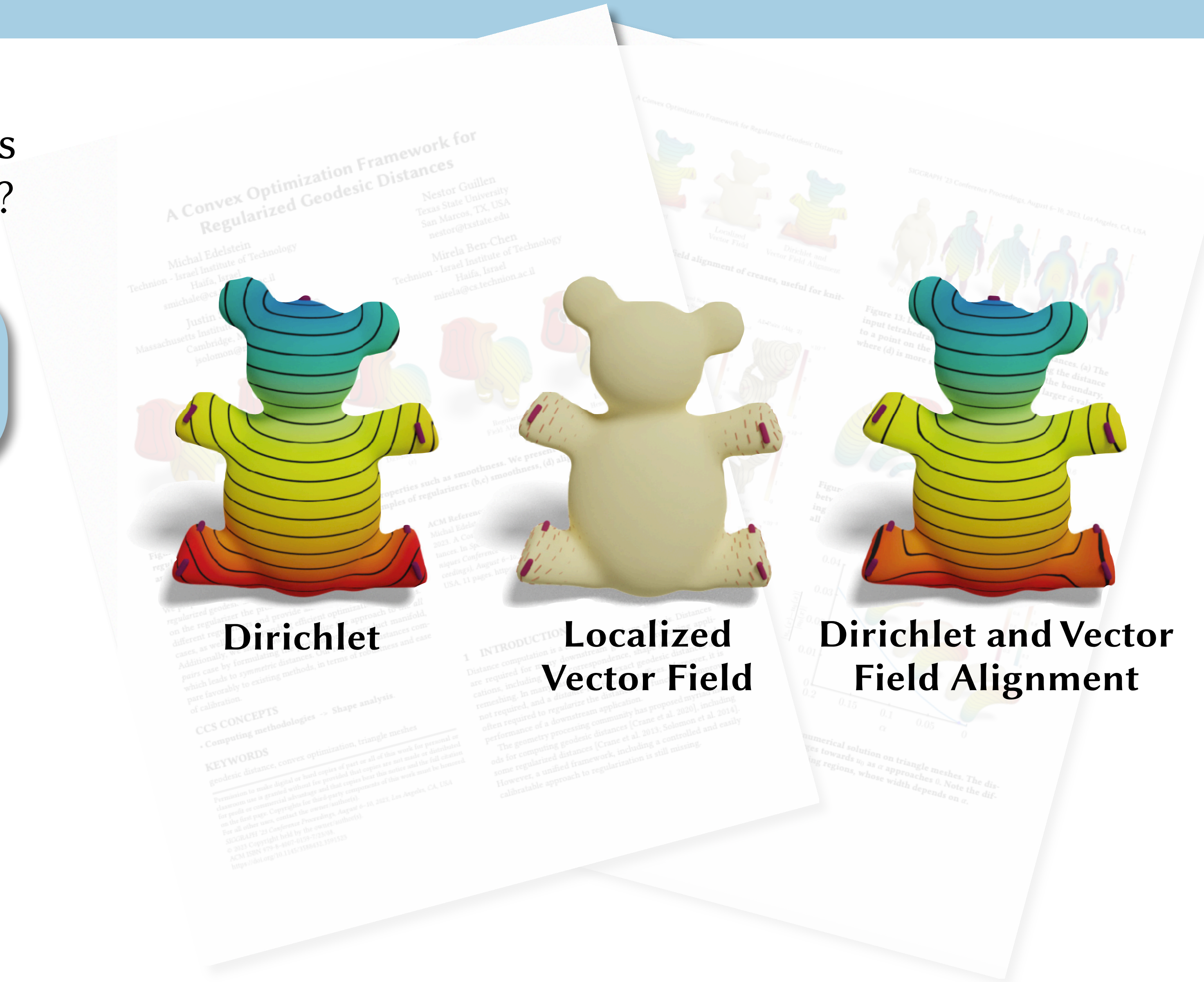
VISCOSITY SOLUTIONS

Question: What if we want the distances to be regularized instead of approximate?

A Convex Optimization Framework for Regularized Geodesic Distances
Edelstein, Guillen, Solomon, Ben-Chen (2023)

Regularizer

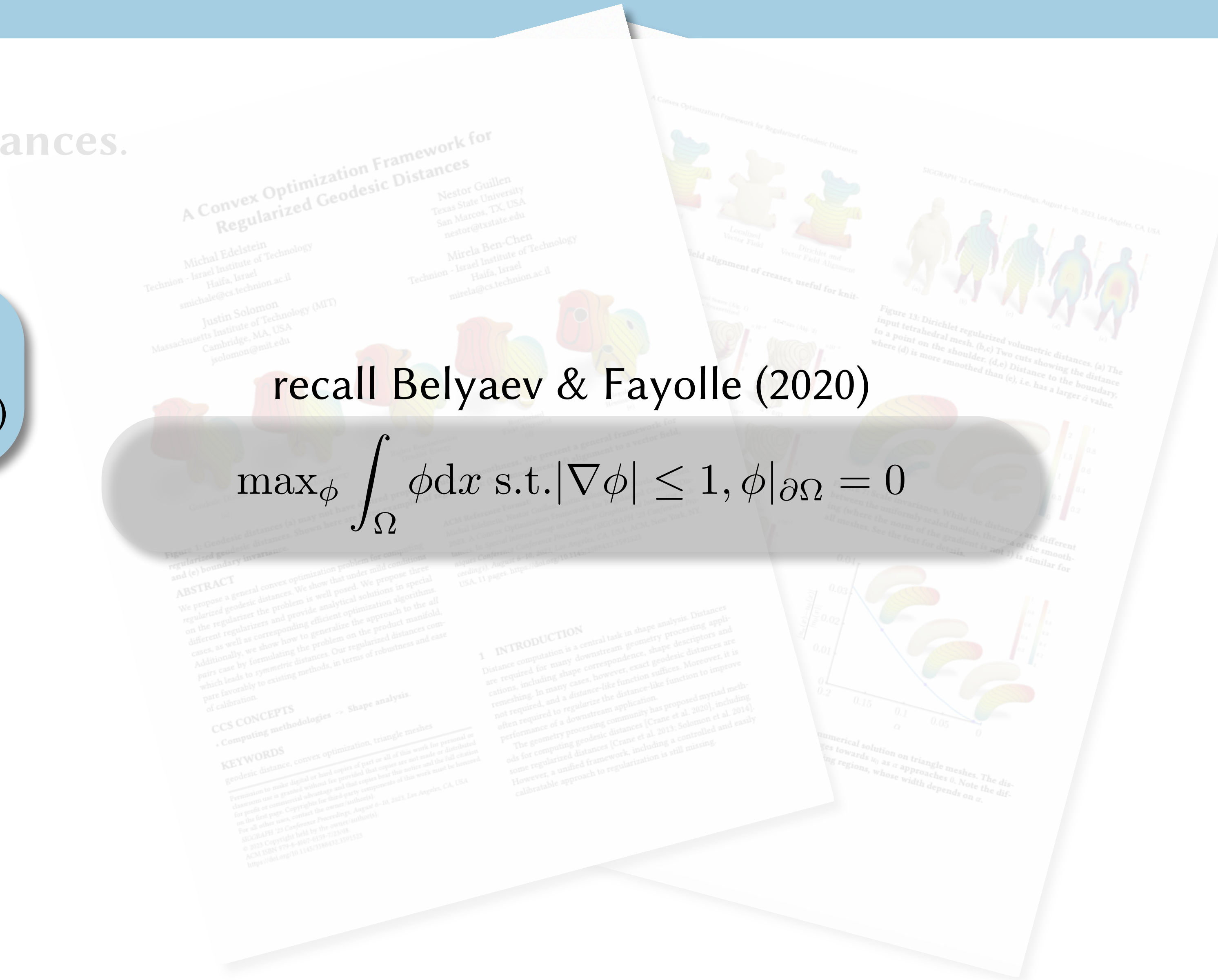
$$\mathcal{E}(u) = \int_{\Omega} F(\nabla u(x), x) \mathrm{d}\,\mathrm{Vol}(x)$$



VISCOSITY SOLUTIONS

Computing regularized geodesic distances.

A Convex Optimization Framework for Regularized Geodesic Distances
Edelstein, Guillen, Solomon, Ben-Chen (2023)



recall Belyaev & Fayolle (2020)

$$\max_{\phi} \int_{\Omega} \phi \mathrm{d}x \text{ s.t.} |\nabla \phi| \leq 1, \phi|_{\partial\Omega} = 0$$

💡 Key Insight

$$\min_u \mathcal{E}(u) - \int_{\mathcal{M}} u \mathrm{d}x$$
$$\text{s.t.} |\nabla u| \leq 1, u|_{\partial\mathcal{M}} = 0$$

Same relaxation strategy: supersolutions

VISCOSITY SOLUTIONS

Question: What if we want to solve more general time-evolving PDE?

A Framework for Solving Parabolic Partial Differential Equations on Discrete Domains

Mattos Da Silva, Stein, Solomon (2024)

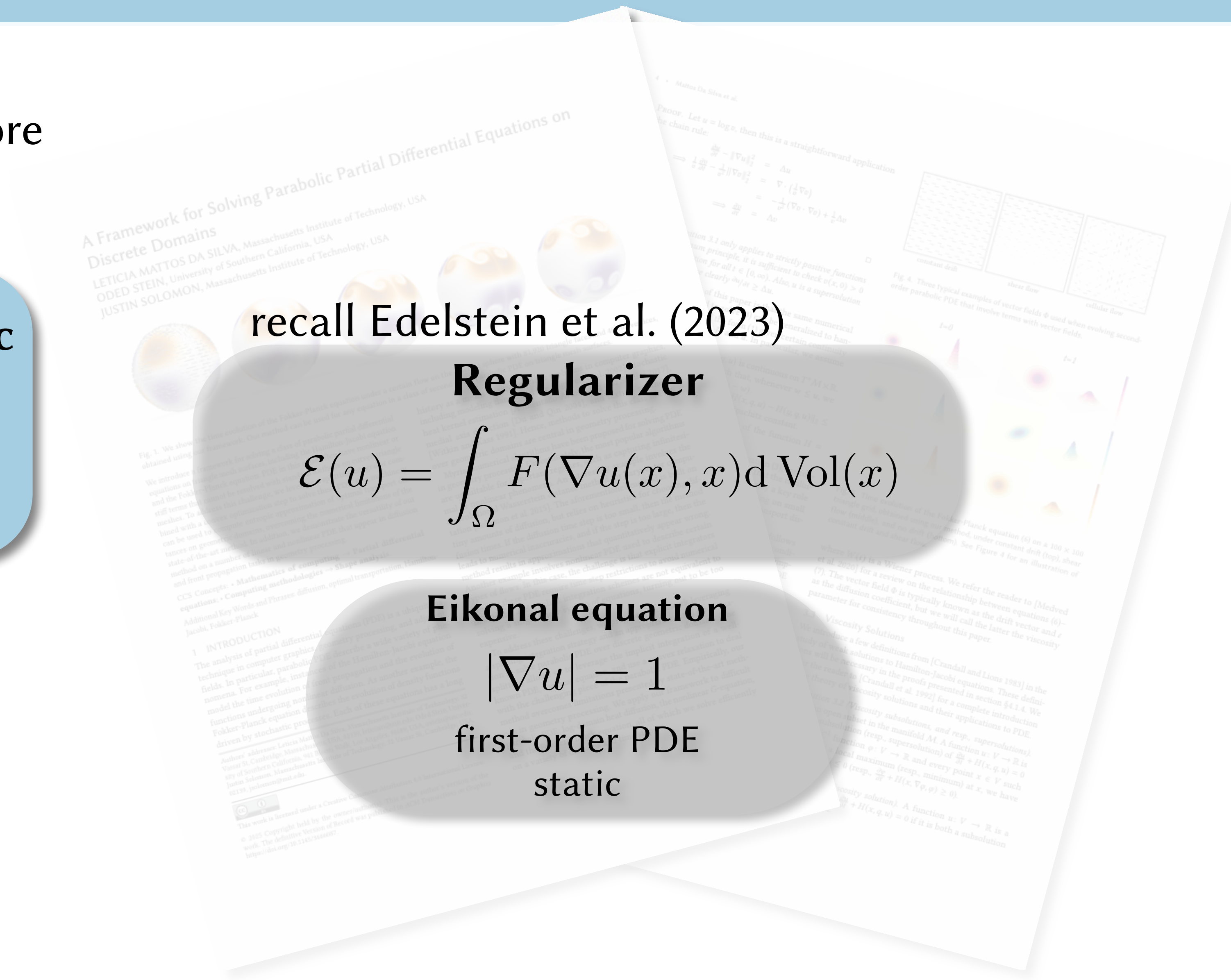
Second-order parabolic PDE

$$\frac{\partial u}{\partial t} + H(x, u, \nabla u) = \varepsilon \Delta u$$


A Framework for Solving Parabolic Partial Differential Equations on Discrete Domains
LETICIA MATTOS DA SILVA, Massachusetts Institute of Technology, USA
ODED STEIN, University of Southern California, USA
JUSTIN SOLOMON, Massachusetts Institute of Technology, USA

VISCOSITY SOLUTIONS

Question: What if we want to solve more general time-evolving PDE?



A Framework for Solving Parabolic Partial Differential Equations on Discrete Domains

Mattos Da Silva, Stein, Solomon (2024)

Second-order parabolic PDE

$$\frac{\partial u}{\partial t} + H(x, u, \nabla u) = \varepsilon \Delta u$$

recall Edelstein et al. (2023)

Regularizer

$$\mathcal{E}(u) = \int_\Omega F(\nabla u(x), x) \mathrm{d}\,\mathrm{Vol}(x)$$

Eikonal equation

$$|\nabla u| = 1$$

first-order PDE

static

VISCOSITY SOLUTIONS

Question: What if we want to solve more general time-evolving PDE?

A Framework for Solving Parabolic Partial Differential Equations on Discrete Domains
Mattos Da Silva, Stein, Solomon (2024)

Second-order parabolic PDE

$$\frac{\partial u}{\partial t} + H(x, u, \nabla u) = \varepsilon \Delta u$$

Same* relaxation strategy: supersolutions

first, apply splitting technique.

1. $\frac{\partial u}{\partial t} = \varepsilon \Delta u$ Linear solve
2. $\frac{\partial u}{\partial t} + H(x, \nabla u, u) = 0$ Convex relaxation
3. $\frac{\partial u}{\partial t} = \varepsilon \Delta u$ Linear solve, again

VISCOSITY SOLUTIONS

Question: What if we want to solve more general time-evolving PDE?

💡 **KEY INSIGHT**

$$\begin{aligned} \arg\min_u \quad & \int_{\mathcal{M}} u(x)\,\mathrm{dVol}(x) \\ \text{subject to} \quad & \tfrac{1}{h}(u - u_n^{(1)}) + H(x, \nabla u, u) \geq 0 \\ & \text{for all } x \in \mathcal{M}. \end{aligned}$$

Second-order parabolic PDE

$$\frac{\partial u}{\partial t} + H(x, u, \nabla u) = \varepsilon \Delta u$$

Same* relaxation strategy: supersolutions

first, apply splitting technique.

1. $\dfrac{\partial u}{\partial t} = \varepsilon \Delta u$ Linear solve

2. $\dfrac{\partial u}{\partial t} + H(x, \nabla u, u) = 0$ Convex relaxation

3. $\dfrac{\partial u}{\partial t} = \varepsilon \Delta u$ Linear solve, again

OUR ROADMAP

OPTIMIZATION

1. The Fundamentals
2. Convexity
3. Why Convexity?
4. Standard Convex Problems

CONVEX RELAXATION

5. Viscosity Solutions
6. Mapping Problems
7. Optimal Transport
8. SOS Relaxations
9. Convex Substructures

MAPPING PROBLEMS

Point-to-point correspondence

Point Registration via Efficient Convex Relaxation

Maron, Dym, Kezurer, Kovalsky, Lipman (2016)

MAPPING PROBLEMS

Relaxing QCQP into SDP-LC

Procrustes matching

$$\text{minimize} \; \|RP - QX\|_F^2$$
$$\text{s.t.} \;\; R^\top R = I, X \; \text{a permutation}$$

💡 **KEY INSIGHT**

non-convex QCQP can be relaxed to SDP at the cost of exactness

MAPPING PROBLEMS

Relaxing QCQP into SDP-LC

Procrustes matching

minimize $\|RP - QX\|_F^2$

s.t. $R^\top R = I, X$ a permutation

💡 **KEY INSIGHT**

non-convex QCQP can be relaxed to SDP at the cost of exactness

lots of steps... let's stick to the big ideas!

introduce new variable

$$Z = zz^\top, \; z = [\text{vec}(R), \text{vec}(X)]$$

and rewrite in standard quadratic

constraints are then linear, but also introduces PSD and *rank(1)* constraints

drop the *rank(1)* constraints

MAPPING PROBLEMS

Question: What if instead of relaxing the Procrustes matching, we relax the requirement that correspondences be exact point-to-point bijections?

💡 **lesson:**

another strategy is to relax the problem *itself* to obtain an already convex formulation instead of performing a proper relaxation.

MAPPING PROBLEMS

Soft Maps Between Surfaces

Solomon, Nguyen, Butscher, Ben-Chen, Guibas (2012)

💡 **KEY INSIGHT**

Replace the "all or nothing" bijection constraints with linear probabilistic ones.

allow mass-splitting

OUR ROADMAP

OPTIMIZATION

1. The Fundamentals
2. Convexity
3. Why Convexity?
4. Standard Convex Problems

CONVEX RELAXATION

5. Viscosity Solutions
6. Mapping Problems
7. Optimal Transport
8. SOS Relaxations
9. Convex Substructures

OPTIMAL TRANSPORT

Optimal transport

Monge's original formulation

$$\int_M \frac{1}{2} c(x, T(x)) \mathrm{d}\mu_0(x)$$

$$\text{s.t. } T_{\#}\mu_0 = \mu_1$$

NON-CONVEX!

🔑 seeking a deterministic map T that maps points to points.

OPTIMAL TRANSPORT

Optimal transport

Kantorovich relaxation

$$\min_{\pi} \int\int \frac{1}{2} c(x,y) \mathrm{d}\pi(x,y)$$

$$\text{s.t. } \pi \in \Pi(\mu_0, \mu_1)$$

💡 seeking a joint probability distribution.

Earth Mover's Distances on Discrete Surfaces
Solomon, Rustamov, Guibas, Butscher (2014)

Convolutional Wasserstein Distances: Efficient Optimal Transportation on Geometric Domains.
Solomon, de Goes, Peyré, Cuturi, Butscher, Nguyen, Du, Guibas (2015)

OPTIMAL TRANSPORT

🔑 change of variable

$m = \mu v$

Optimal transport

Benamou-Brenier

$$\min_{\mu_t, v_t} \int_0^1 \int_M \frac{1}{2} \|v_t(x)\|^2 \mathrm{d}\mu_t(x)\mathrm{d}t$$

$$\text{s.t.} \quad \partial \mu_t + \nabla \cdot (\mu v) = 0$$

$$\mu(0) = \mu_0, \mu(1) = \mu_1$$

NON-CONVEX!

$$\min_{\mu_t, m_t} \int_0^1 \int_M \frac{1}{2} \frac{\|m_t(x)\|^2}{\mu_t} \mathrm{d}\mu_t(x)\mathrm{d}t$$

$$\text{s.t.} \quad \partial \mu_t + \nabla \cdot m = 0$$

$$\mu(0) = \mu_0, \mu(1) = \mu_1$$

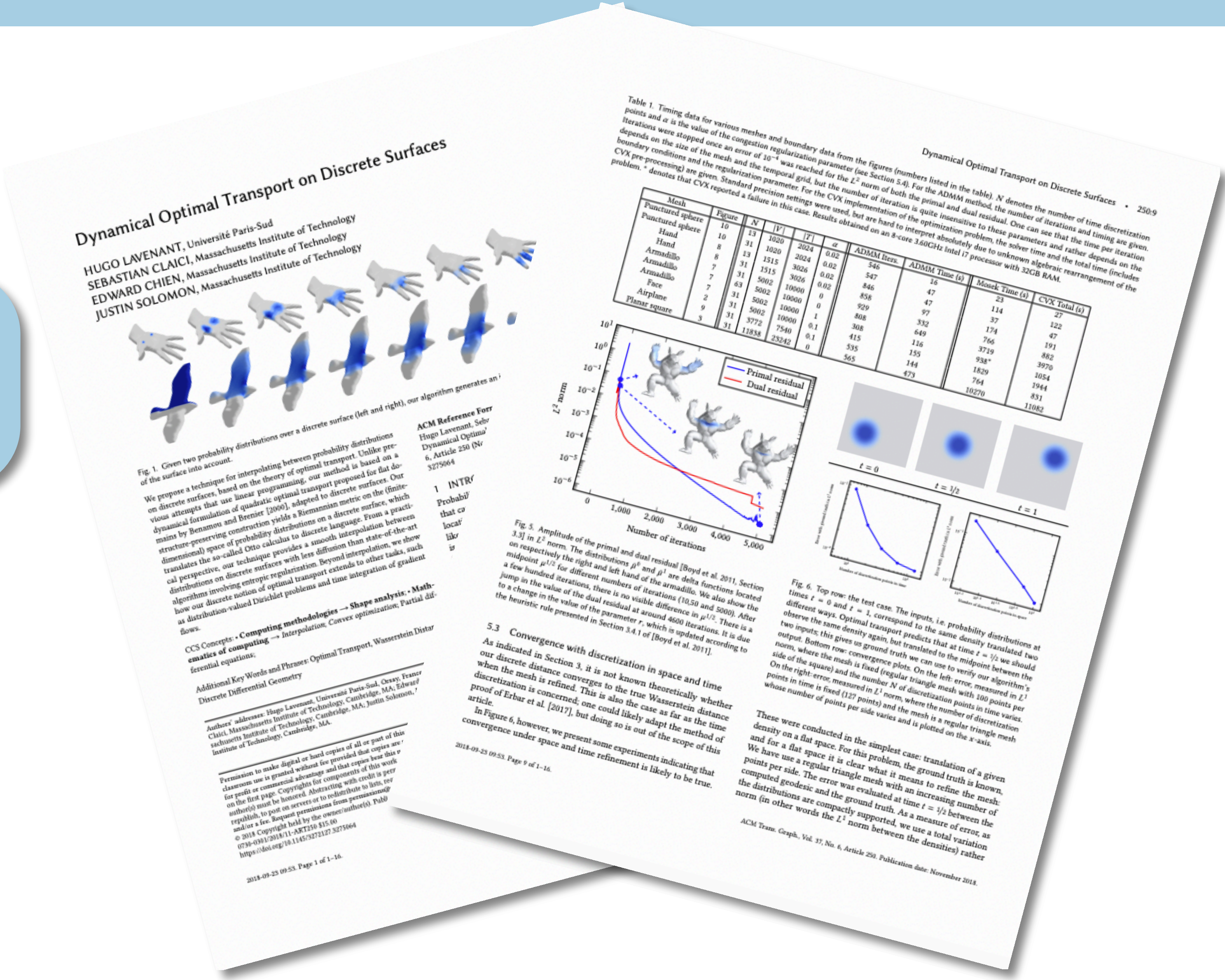
CONVEX

OPTIMAL TRANSPORT

Optimal transport

Dynamical Optimal Transport on Discrete Surfaces

Lavenant, Claici, Chien, Solomon



OUR ROADMAP

OPTIMIZATION

1. The Fundamentals
2. Convexity
3. Why Convexity?
4. Standard Convex Problems

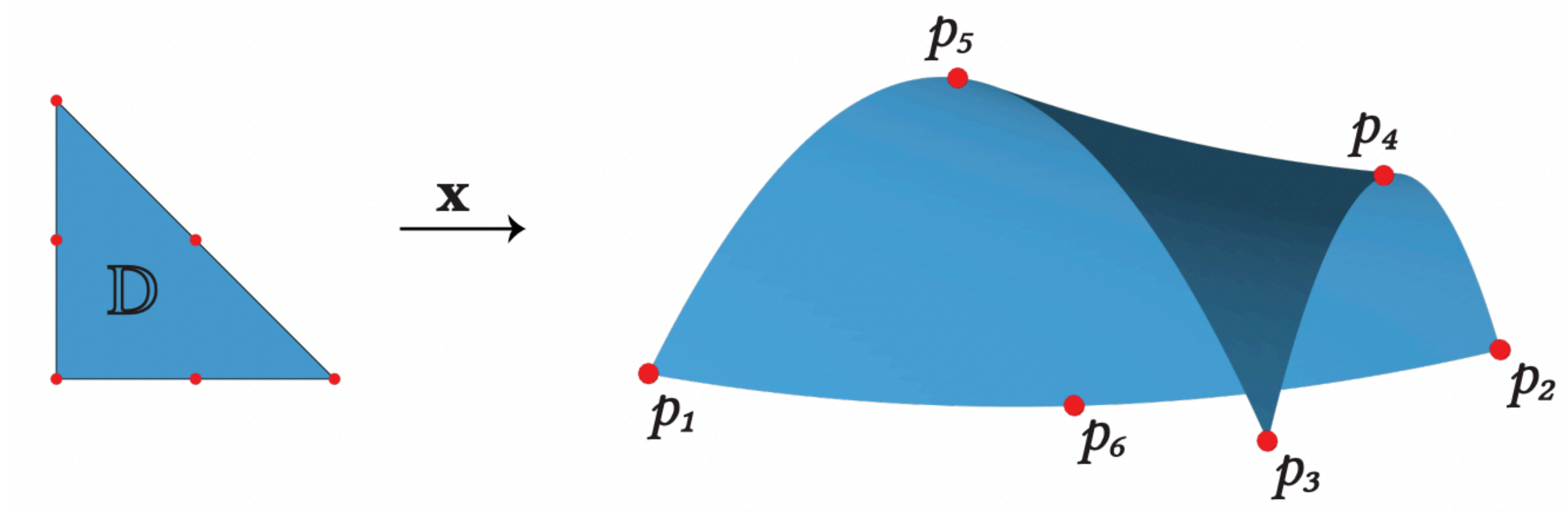
CONVEX RELAXATION

5. Viscosity Solutions
6. Mapping Problems
7. Optimal Transport
8. SOS Relaxations
9. Convex Substructures

SIMPLIFIED TAXONOMY OF CONVEX

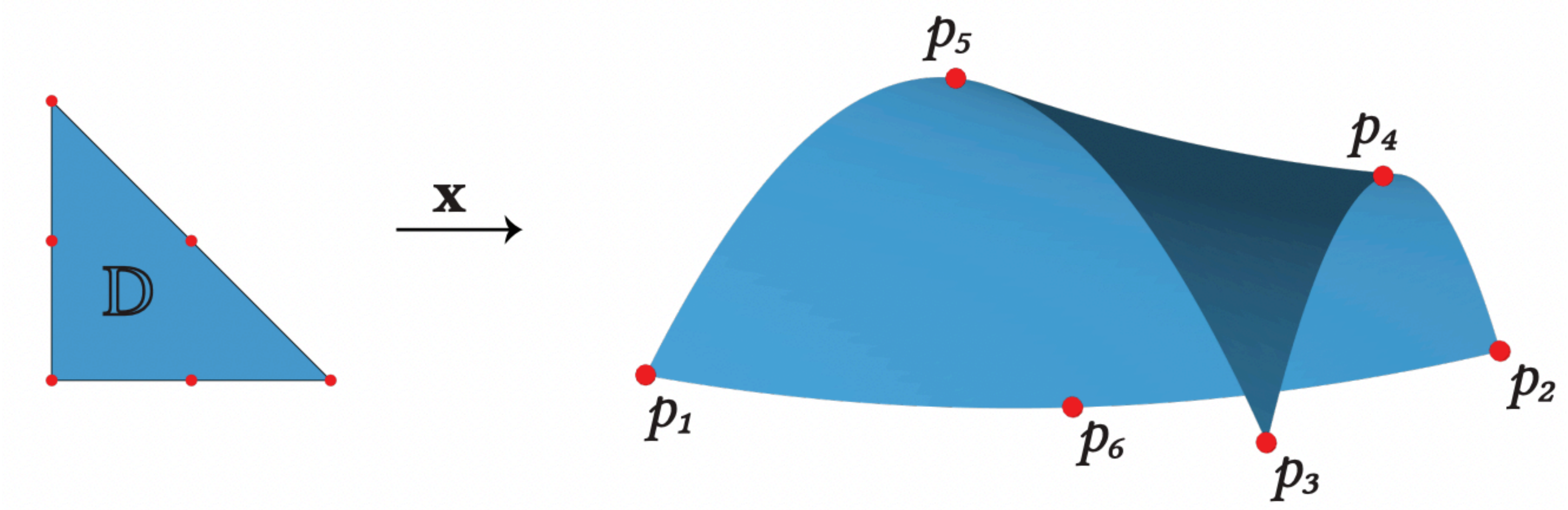
SOS RELAXATION

↩️ **This will connect back to SDP relaxation**



SOS RELAXATION

↩️ **This will connect back to SDP relaxation**



Putinar's Positivity Theorem

If f is polynomial, then f being nonnegative on $\mathbb{D}$ is equivalent to f being expressible as a sum of squares of polynomials.

SOS RELAXATION

equivalent to

Typical Optimization Problem

$$\min_{x \in \mathbb{D}} f(x)$$

Reformulation with Linear Objective

$$\begin{aligned} &\max \ \lambda \\ &\text{s.t.} \ \ f(x) - \lambda \geq 0 \\ &\qquad x \in \mathbb{D} \end{aligned}$$

Putinar's Positivity Theorem

If f is polynomial, then f being nonnegative on $\mathbb{D}$ is equivalent to f being expressible as a sum of squares of polynomials.

SOS RELAXATION

Typical Optimization Problem

$$\min_{x \in \mathbb{D}} f(x)$$

Reformulation with Linear Objective

$$\begin{aligned} &\max \ \lambda \\ &\text{s.t.} \ \ f(x) - \lambda \geq 0 \\ &\qquad x \in \mathbb{D} \end{aligned}$$

Putinar's Positivity Theorem

If f is polynomial, then f being nonnegative on $\mathbb{D}$ is equivalent to f being expressible as a sum of squares of polynomials.

Can solve via SOS relaxation if:

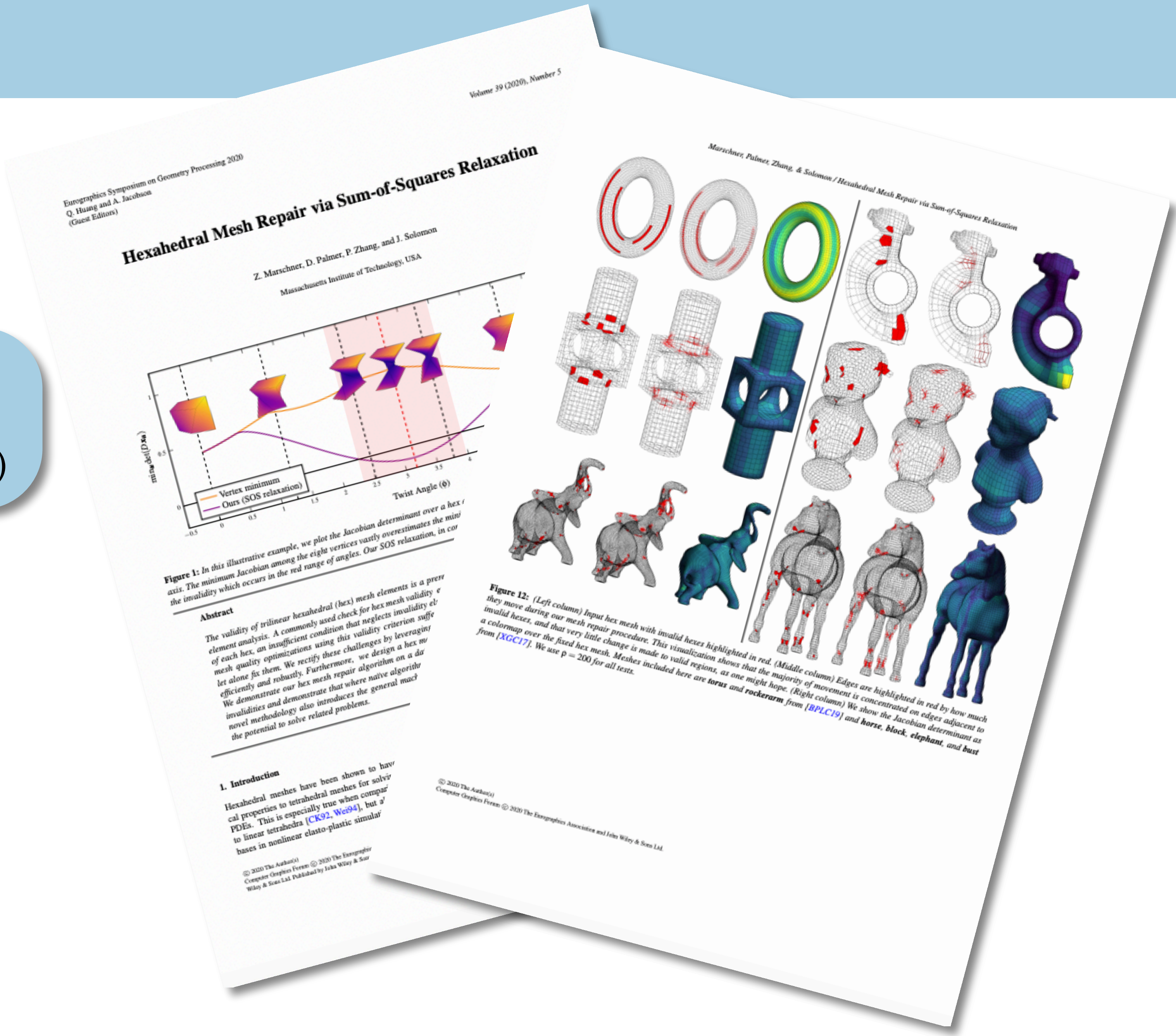
- domain compact with polynomial boundary
- objective is polynomial function

Solve resulting problem via SDP relaxation!

SOS RELAXATION

Mesh repair

Hexahedral Mesh Repair via
Sum-of-Squares Relaxation
Marschner, Palmer, Zhang, Solomon (2020)



OUR ROADMAP

OPTIMIZATION

1. The Fundamentals
2. Convexity
3. Why Convexity?
4. Standard Convex Problems

CONVEX RELAXATION

5. Viscosity Solutions
6. Mapping Problems
7. Optimal Transport
8. SOS Relaxations
9. Convex Substructures

CONVEX SUBSTRUCTURES

💡 **lesson:**

A number of optimization problems cannot be recast as fully convex. A *hidden convexity* can at least isolate the non-convex part. While this strategy is not a proper convex relaxation, it can lead to optimization algorithms that enjoy some of the benefits of convex optimization.

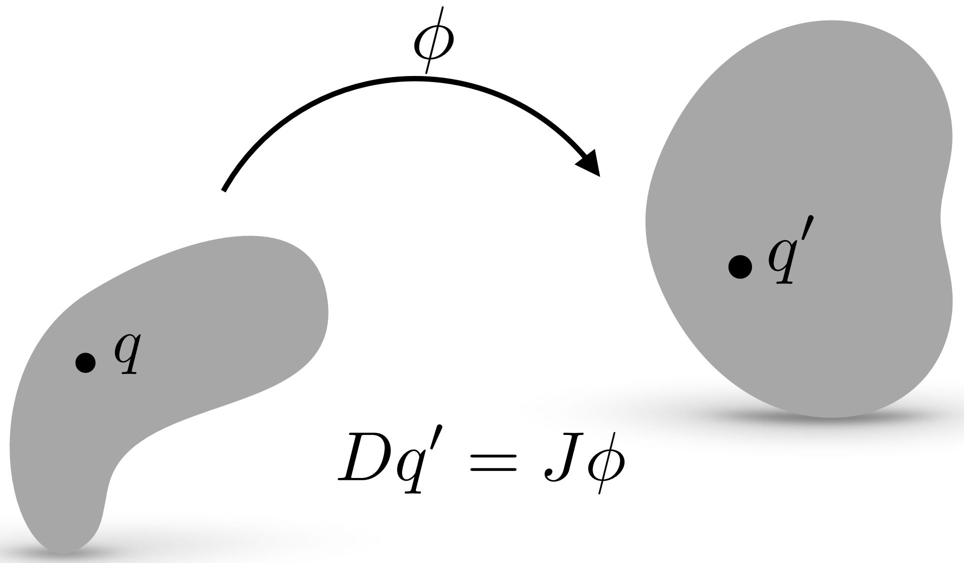
CONVEX SUBSTRUCTURES

🚧 **Identifying convex substructure in distortion energies**

Distortion energy

$$E(q) := \sum_i w_i f((Dq)_i)$$

NON-CONVEX!

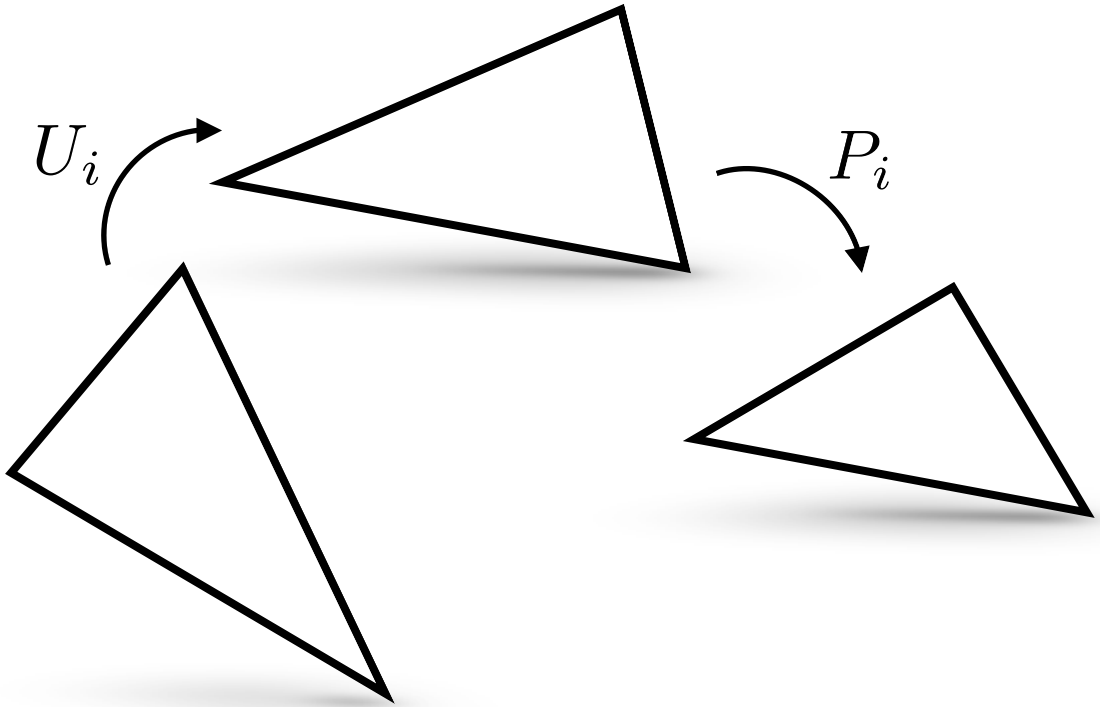


Deformation gradient

CONVEX SUBSTRUCTURES

Apply polar decomposition to each Jacobian matrix $J_i = U_i P_i$

$$\min_{U \in SO(d), P \in \mathcal{S}_+^d} E(Dq) = \sum_i w_i f(P_i)$$
$$\text{s.t. } (Dq)_i - U_i P_i = 0$$



A Splitting Scheme for Flip-Free Distortion Energies

Oded Stein*, Jiajin Li†, and Justin Solomon‡

Abstract. We introduce a robust optimization method for flip-free distortion energies used, for example, in parametrization, deformation, and volume correspondence. This method can minimize a variety of distortion energies, such as the symmetric Dirichlet energy and our new symmetric gradient energy. We identify and exploit the special structure of distortion energies to employ an operator splitting technique, leading us to propose a novel **A**lternating **D**irection **M**ethod of **M**ultipliers (ADMM) algorithm to deal with the non-convex, non-smooth nature of distortion energies. The scheme results in an efficient method where the global step involves a single matrix multiplication and the local steps are closed-form per-triangle/per-tetrahedron expressions that are highly parallelizable. The resulting general-purpose optimization algorithm exhibits robustness to flipped triangles and tetrahedra in initial data as well as during the optimization. We establish the convergence of our proposed algorithm under certain conditions and demonstrate applications to parametrization, deformation, and volume correspondence.

Key words. computer graphics, optimization, nonconvex optimization, parametrization, ADMM

AMS subject classifications. 65K10, 90C26, 65D18, 68U05

1. Introduction. Distortion energies measure how much a mapping from one shape to another deforms the initial shape. Minimizing these energies can yield maps between domains with as little distortion as possible. Minimization of distortion energies with a variety of constraints is employed in a wide array of computer graphics applications, such as UV mapping (where one seeks to embed a 3D surface into 2D while minimizing distortion, Figure 1 *left*), deformation (where parts of a surface or volume are deformed, and the goal is to find the overall deformation with least distortion, Figure 1 *center*), and volume correspondence (two boundary surfaces are given, and a distortion-minimizing map between the two volumes is desired, Figure 1 *right*). We are interested in computing maps that minimize distortion energies on triangle and tetrahedral meshes.

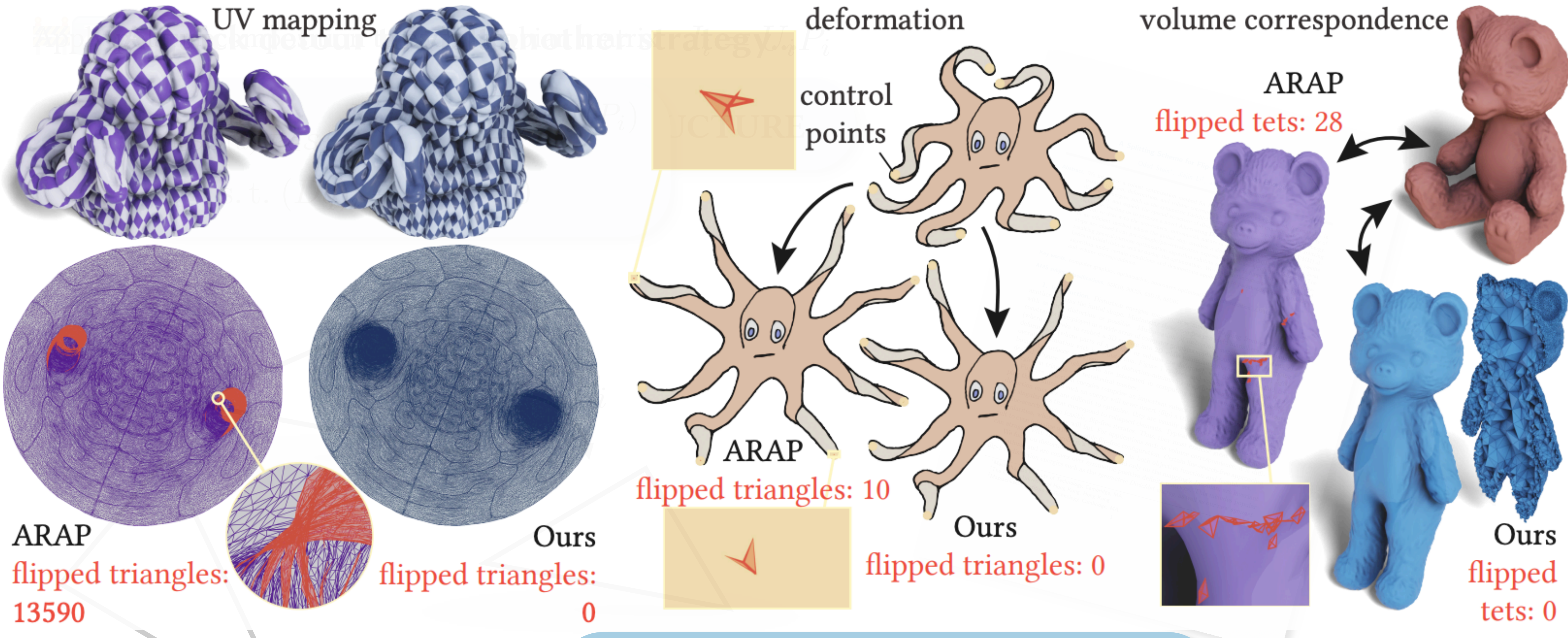
Flip-free distortion energies comprise an important subset of distortion energies. A mapping that minimizes such an energy will never invert (flip) a triangle or tetrahedron. Flip-free distortion energies are difficult to optimize: they are usually non-linear and non-convex, and have singularities that correspond to collapsed elements. Typical optimization methods based on line-search require feasible, flip-free iterates. Thus, they must exercise great care to avoid singularities, where they will fail. For applications such as volume correspondence, it is difficult to even initialize with a feasible flip-free configuration. Certain line-search-free approaches can struggle with the problem's non-convexity in both objective function and feasible set.

We focus on distortion energies that depend only on the mapping's Jacobian, are invariant to rotations, and are convex over symmetric positive definite matrices. This includes popular flip-free distortion energies such as the *symmetric Dirichlet energy*, as well as our new *symmet-*

*Massachusetts Institute of Technology, Cambridge, MA.
†The Chinese University of Hong Kong, Hong Kong.
‡Massachusetts Institute of Technology, Cambridge, MA.

Stein, Li, Solomon (2021)

CONVEX SUBSTRUCTURES



many possible applications!

Stein, Li, Solomon (2021)

CONVEX SUBSTRUCTURES

Question: What happens if we add dynamics?

Lagrangian Mechanics

convex → K, concave → E

$$\min \mathcal{L}, \quad \mathcal{L} = K - E$$

the solution of this optimization problem gives the equations of motion

Not obvious that the hidden convexity will be helpful!

CONVEX SUBSTRUCTURES

Question: What happens if we add dynamics?

Lagrangian Mechanics

convex → K, concave → E

$$\min \mathcal{L}, \quad \mathcal{L} = K - E$$

the solution of this optimization problem gives the equations of motion

Not obvious that the hidden convexity will be helpful!

Mostly Convex Reformulation

convex → K', hidden convexity → E'

$$\min K' + E'$$

Preserve first-order optimality constraints!

simplified preview (not the full story)

CONVEX SUBSTRUCTURES

Question: What happens if we add dynar

Lagrangian Mechanics

Variational Elastodynamic Simulation

Mattos Da Silva, Sellán, Pacheco-Tallaj, Solomon (2025)

Solution gives equations of motion

Not obvious that the hidden convexity will be helpful!


Variational Elastodynamic Simulation

LETICIA MATTOS DA SILVA, Massachusetts Institute of Technology
SILVIA SELLÁN, Massachusetts Institute of Technology
NATALIA PACHECO-TALLAJ, Massachusetts Institute of Technology
JUSTIN SOLOMON, Massachusetts Institute of Technology

1 INTRODUCTION

4.1 Hidden Convexity in the Action Functional

OUR ROADMAP

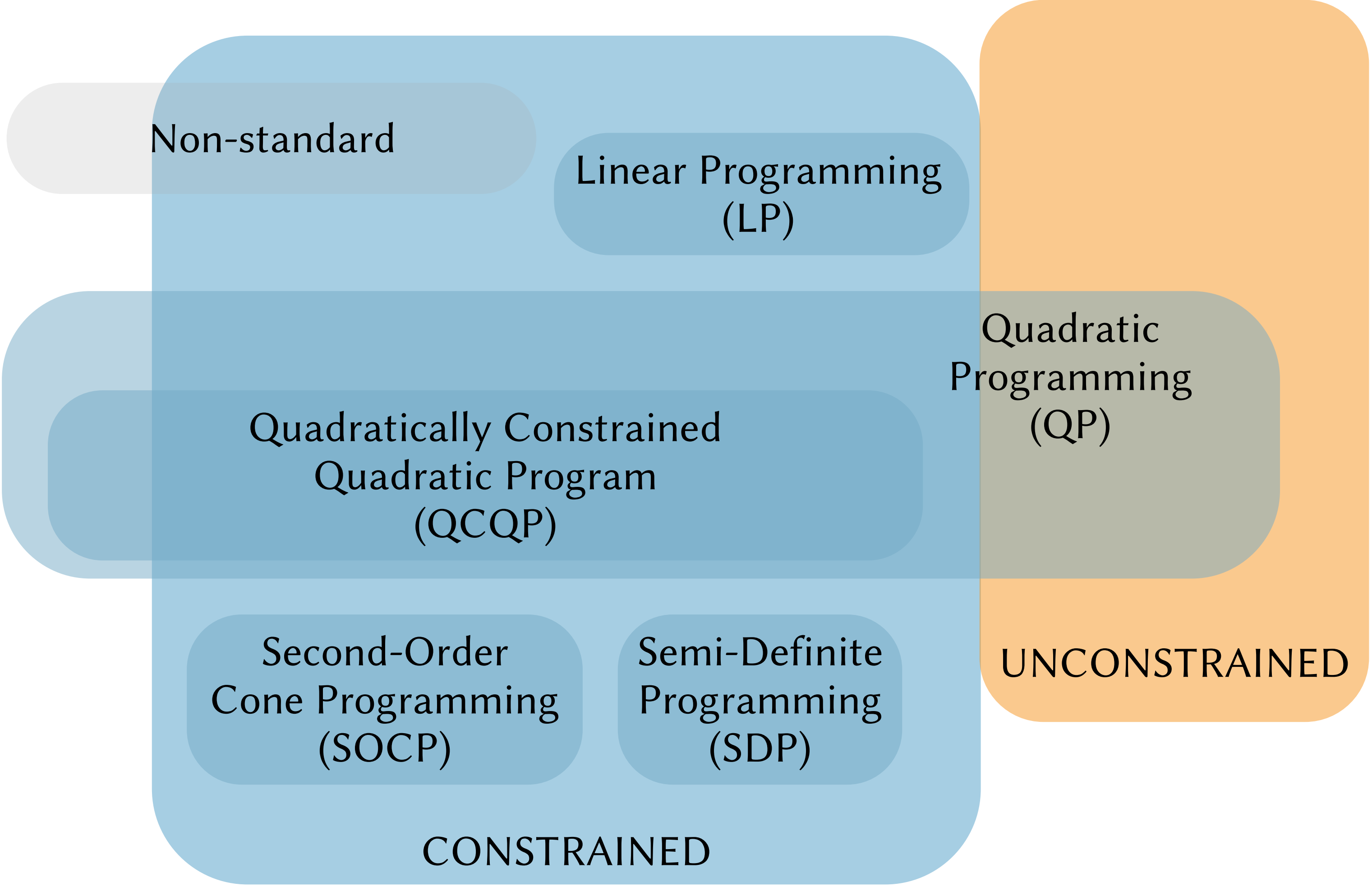
OPTIMIZATION

1. The Fundamentals
2. Convexity
3. Why Convexity?
4. Standard Convex Problems

CONVEX RELAXATION

5. Viscosity Solutions
6. Mapping Problems
7. Optimal Transport
8. SOS Relaxations
9. Convex Substructures

SIMPLIFIED TAXONOMY OF CONVEX



SUGGESTED READING

OPTIMIZATION

Snakes: Active Contour Models

Kass, Witkin and Terzopoulos (1988)

STANDARD CONVEX PROBLEMS

Bounded Biharmonic Weights for Real-Time Deformation

Jacobson, Baran, Popović, Sorkine (2011)

Controlling Singular Values with Semidefinite Programming

Kovalsky, Aigerman, Basri, Lipman (2014)

RELATED READING

Convex Optimization

Boyd & Vandenberghe (2004)

SUGGESTED READING

VISCOSITY SOLUTIONS

An ADMM-based Scheme for Distance Function Approximation
Belyaev & Fayolle (2020)
A Convex Optimization Framework for Regularized Geodesic Distances
Edelstein, Guillen, Solomon, Ben-Chen (2023)
A Framework for Solving Parabolic Partial Differential Equations on Discrete Domains
Mattos Da Silva, Stein, Solomon (2024)

RELATED READING

Geodesics in Heat : A New Approach to Computing Distance Based on Heat Flow
Crane, Weischedel, Wardetzky (2013)
The Discrete Geodesic Problem
Mitchell, Mount, Papadimitriou (1987)
Fast Exact and Approximate Geodesics on Meshes
Surazhsky, Surazhsky, Kirsanov, Gortler, Hoppe (2005)

SUGGESTED READING

MAPPING PROBLEMS

Point Registration via Efficient Convex Relaxation
Maron, Dym, Kezurer, Kovalsky, Lipman (2016)

Soft Maps Between Surfaces
Solomon, Nguyen, Butscher, Ben-Chen, Guibas (2012)

RELATED READING

Blended Intrinsic Maps
Kim, Lipman, Funkhouser (2011)

OPTIMAL TRANSPORT

Earth Mover's Distances on Discrete Surfaces
Solomon, Rustamov, Guibas, Butscher (2014)

Convolutional Wasserstein Distances
Solomon, de Goes, Peyré, Cuturi, Butscher, Nguyen, Du, Guibas (2015)

Dynamical Optimal Transport on Discrete Surfaces
Lavenant, Claici, Chien, Solomon

SUGGESTED READING

SOS RELAXATIONS

Hexahedral Mesh Repair via Sum-of-Squares Relaxation
Marschner, Palmer, Zhang, Solomon (2020)

RELATED READING

Sum-of-Squares Geometry Processing
Marshner, Zhang, Palmer, Solomon (2021)

Sum-of-Squares Collision Detection for Curved Shapes and Paths
Marschner*, Zhang*, Solomon, Tamstorf (2023)

CONVEX SUBSTRUCTURES

A Splitting Scheme for Flip-Free Distortion Energies
Stein, Li, Solomon (2022)

Variational Elastodynamic Simulation
Mattos Da Silva, Sellán, Pacheco-Tallaj, Solomon (2025)

THANK YOU FOR ATTENDING!

Questions?

Presenter's website